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WKB Special Expansion and the Solutions of Neo-Hookean and Varga Elastic Cylinder, Sphere and Cube

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ABSTRACT

In this paper, our main objective is to apply the following specific form of WKB expansion i.e.

$$F(r) = \left(F_0(r) + \frac{1}{n}F_1(r) + \frac{1}{n^2}F_2(r) + \frac{1}{n^3}F_3(r) + \cdots\right) exp(n\int S(r)dr),$$

to the solution of the eigenvalue problems of Varga and Neo-Hookean elastic cylinder, everted cylinder, sphere, everted sphere and cube. The governing equations first formulated as a fourth order eigenvalue problems and then the analytic findings are compared with the counterpart numerical solutions. By applying this form of WKB expansion, we are showing that at the leading order the roots of $S(r)$ are repeated for Varga but un-repeated for Neo-Hookean materials. We also conclude that the differential equations of $F_0(r)$, $F_1(r)$, $F_2(r)$, ... become second order for Varga and first order for Neo-Hookean materials.

Keywords: WKB method, Eversion, Buckling, Incompressible, Varga and Neo-Hookean materials.

INTRODUCTION

For the first time Wilkes [1], considered the stability of thick-walled solids, who found conditions for the axially symmetric buckling of a tube under end thrust. Nowinski and Shahinpoor [2], Wang and Ertepinar [3], Sierakowski, Sun and Ebcioglu [4] and Hill [5,6], all have studied circular cylindrical shells under external pressure and also Sanjaranipour [7,8] applied WKB method on solving this problem. Patterson [9], Haughton and Ogden [10,11] and Zhu, Luo and Ogden [12] examined the circular cylindrical tubes when axial compression is combined with the internal or external pressure. The problem of everting incompressible isotropic hyperelastic right circular cylinders has been considered both experimentally and theoretically by Rivlin [13], Chadwick and Haddon [14], Chadwick [15], Adeleke [16], Truesdell [17]), Chen and Haughton [18]. This eigen-value problem has been solved also by Haughton and Orr [19] with the use of the compound matrix method. Fu and Sanjaranipour [20] and Fu and Lin [21] applied WKB method on solving this problem for Varga and Neo-Hookean materials, respectively. New and better methods introduced for the first time on pure bending by Triantafyllidis [22]. Haughton [23] did a similar analysis for the hyperelastic materials in a three-dimensional context and also discussed the vertical compression. Coman and Destrade [24] studied the deformation of the pure bending of a rubber block in plane strain and employed WKB method on solving the fourth order ODE,s of this problem. Destrade, Annaidh and Coman [25] described the critical stretch ratio of a bent block for several models. Recently Sanjaranipour, Hatami and Abdolalian [26] applied WKB method with repeated roots to study the bifurcation of the pure bending of a rubber block made of an elastic Varga material and described the angle of bending and the azimuthal shear. The buckling analysis of a spherical shell under uniform external pressure has been studied by Weslowski [27] and Hill [28] and also Fu [29] applied the standard form of WKB expansion (See, Bender and Orszag [30]), for the stability analysis of this problem. Bifurcation of spherical elastic shells for a finite radially symmetric inflation are analyzed by Haughton and Ogden [31]. The problem of existence and uniqueness of the solutions of an everted spherical shell was investigated by Ericksen [32], Antman and Srubshchink [33] and Liu [34]. It should be emphasized that this problem also has been derived and solved with the aid of compound matrix and WKB methods by Haughton and Chen [35,36].

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Indeed since most of the eigen-value problems of this paper have been studied in different articles and different aspects have been analyzed with different views and/or written for special purposes, there is no need of presenting such common relations again and we have just urged on such parts of the derivations, which are really needed and are not considered on those articles. Our main concern in this research is to find, solve and compare the differential equations related to $F_0(r)$, $F_1(r)$, $F_2(r)$, ... of the WKB expansion. Another concerns is to study the situation of $S(r)$ of this expansion. To demonstrate this matter, the fourth order ODE,s of the eigenvalue problems related to cylinder, everted cylinder, cube, sphere and everted sphere for Varga and Neo-Hookean materials is used. To illustrate this matter, first we focus our attention to the physical and the geometry of the mentioned problems and later the equilibrium and the governing equations will be written down. The WKB solution for each of the cases will be discuss and then at the end of each section, the results obtained will be compared with the results of the compound matrix method. In order to be able to compare the different scenarios, most of the problems have been resolved once again by the use of the compound matrix method. It should be emphasized that for applying the WKB method, the symbolic Mathematica software and for numerical compound matrix method, Fortran 90 programming has been employed.

Cylindrical shell

In this section, we have given a brief description of the geometric definitions and the derived governing equation of the cylinder including everted cylinder by Haughton and Ogden [11] and Haughton and Orr [19].

The buckling of a cylindrical shell

An incompressible isotropic homogeneous elastic shell has been considered. The undeformed cylindrical shell is defined by

$$A \leq R \leq B, 0 \leq \Theta \leq 2\pi, 0 \leq Z \leq L, \quad (1)$$

where (R, Θ, Z) are the cylindrical polar coordinates and L is the length of the tube. For the problems of this and the next sections, we shall assume that all the variables and parameters which have dimension of length have been scaled by B. The cylinder is subjected to an external hydrostatic pressure on its outer surface so that the deformed cylindrical shell occupying the region

$$a \leq r \leq b, 0 \leq \theta \leq 2\pi, 0 \leq z \leq l, \quad (2)$$

where (r, θ, z) are also the cylindrical polar coordinates in the deformed configuration. The deformation now is assumed to be

$$r = r(R), \theta = \Theta, \qquad z = \lambda_z Z, \quad (3)$$

where λ_z is the axial stretch. The principal stretches of the deformation can be written by

$$\lambda_1 = \lambda^{-1}, \lambda_2 = \frac{r}{R} = \lambda, \lambda_3 = \lambda_z = 1. \quad (4)$$

In this article $(1,2,3) \equiv (r, \theta, z)$. In view of $(4)_2$, the deformed inner and outer radii are defined by $a = \lambda_a A$ and $b = \lambda_b$, where λ_a and λ_b are constants which for the present problem satisfy $(0 < \lambda_a, \lambda_b < 1)$. With the use of (4) and the incompressible condition $\det(F) = \lambda_1\lambda_2\lambda_3 = 1$, we have

$$r^2 = R^2 - A^2 + a^2, \lambda_b^2 = 1 + A^2(\lambda_a^2 - 1), \quad (5)$$

where F is the deformation gradient. For the symmetric configuration considered here, the only equilibrium equation not satisfied trivially is

$$r\frac{d\sigma_{11}}{dr} + \sigma_{11} - \sigma_{22} = 0, \quad (6)$$

and the principal Cauchy stresses are given by

$$\sigma_{ii} = \sigma_i - p = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, (i = 1,2,3) \quad (7)$$

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where p is the hydrostatic pressure that arises due to the incompressible condition. The strain-energy function $W = W(\lambda_1, \lambda_2, \lambda_3)$ for Neo-Hookean and Varga materials, respectively are defined by Ogden [37] as

$$W(\lambda_1,\lambda_2,\lambda_3) = \mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3),$$
$$W(\lambda_1,\lambda_2,\lambda_3) = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3), \quad (8)$$

without loss of generality, we assume that μ is $\frac{1}{2}$. For the mentioned deformation, the boundary conditions are

$$\sigma_{11}(a) = 0, \sigma_{11}(b) = -P, \quad (9)$$

where P denotes the external hydrostatic pressure.

The buckling of an everted cylinder

The undeformed incompressible isotropic homogeneous elastic cylindrical shell is defined by Eq. (1). The cylinder is everted into another right circular cylinder that occupying the region

$$a \le r \le b, 0 \le \theta \le 2\pi, \qquad -l \le z \le 0, \quad (10)$$

where (r, θ, z) are the cylindrical polar coordinates. We have to note that in particular case the surface $R = A$ is mapped into the surface $r = b$ and $R = B$ is mapped to $r = a$. The deformation now can be described by

$$r = r(R), \theta = \Theta, \qquad z = -\lambda_z Z, \quad (11)$$

where $r = r(R)$ is a monotonically increasing function. The principal stretches of the deformation here are

$$\lambda_1 = -\frac{dr}{dR}, \lambda_2 = \frac{r}{R} = \lambda, \qquad \lambda_3 = \lambda_z. \quad (12)$$

Thus, with the use of Eq. (12), the deformed inner and outer radii can now be written by $a = \lambda_a$ and $b = \lambda_b\, A$, where λ_a and λ_b are constants which can be taken as the controlling parameter. By using the incompressible condition and Eq. (12), we obtain

$$r^2 = a^2 + \frac{1-R^2}{\lambda_z}, \lambda_b^2 = (1 - A^2 + \lambda_z a^2)/(\lambda_z A^2). \quad (13)$$

For the mentioned deformation, the equilibrium equation Eq. (6) is satisfied. The sides of the cylinder are assumed to be traction free, so

$$\sigma_{11}(a) = \sigma_{11}(b) = 0. \quad (14)$$

We assume that the resultant axial load on any cross-section is zero, then

$$\int_a^b r\, \sigma_{33}\, dr = 0. \quad (15)$$

The eversion problem now is fully defined i.e. in view of the relations Eq. (6), Eq. (14), Eq. (15) and regarding to the explanations given by Fu and Sanjaranipour [20] and Fu and Lin [21], the final eversion conditions for Varga and Neo-Hookean materials, respectively are

$$2\lambda_z^3(b^2 - a^2) - (1 + a^2\lambda_z)(\frac{1}{a} - \frac{A}{b}) = 0, \quad \frac{\sqrt{\lambda_z}(a\,A - b)}{1 + a^2\lambda_z} - \sin(\frac{a\,A - b}{2\,a\,b\,\sqrt{\lambda_z}}) = 0, \quad (16)$$

$$(\frac{1 + a^2\lambda_z}{a^2\lambda_z})(\frac{b^2 - a^2}{b^2}) + 2\ln\frac{a\,A}{b} = 0, \qquad (b^2 - a^2)(\lambda_z + \lambda_z^4) + (1 + \lambda_z a^2)\ln\frac{a\,A}{b} = 0, \quad (17)$$

where, $A = \sqrt{1 - \lambda_z(b^2 - a^2)}$.

Bifurcation criterion

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We write down here a brief description of the equilibrium equations and the boundary conditions by using the derivations of Haughton and Ogden [11] and Haughton and Orr [19] and the common derivations in the later sections. In the absence of the body forces the incremental equilibrium equations are

$$div\, \dot{s} = 0, \tag{18}$$

where div is the divergence operator and $\dot{s}$ is the increment of the nominal stress both in the current configuration. The incremental boundary conditions in respect of a hydrostatic pressure loading are

$$\dot{s}^T\, n = P\ \Gamma\, n - \dot{P}\, n, \tag{19}$$

where n is the unit outward normal to the surface. We should note that P is not equal to zero for the problems with the external hydrostatic pressure and is zero for the everted problem. The incremental constitutive law is

$$\dot{s} = B\ \Gamma + p\ \Gamma - \dot{p}\, I, \tag{20}$$

where I and Γ are respectively the identity tensor and the incremental deformation gradient and B is the fourth order tensor of instantaneous moduli in the current configuration. The non-zero components of B on the principal axes of the underlying deformation are given by Haughton and Ogden [31] and therefore are not repeated here. The incremental displacement is defined by $\dot{x} = (u(r,\theta,z), v(r,\theta,z), w(r,\theta,z))$. Although we shall study the eigen-value problem including the fourth order ODE,s, the case where u, v and w are independent of z also will be considered. Then $\dot{x} = (u(r,\theta), v(r,\theta), 0)$ and the components of Γ are displayed as

$$\Gamma = \begin{bmatrix} u_r & (u_\theta - v)/r & 0 \\ v_r & (u + v_\theta)/r & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{21}$$

where subscripts here denote the partial derivatives. Since the material is incompressible, we have

$$tr\ \Gamma \equiv u_r + (u + v_\theta)/r = 0. \tag{22}$$

With the use of relations Eqs. (18)-(22) and in view of u, v and w which are independent of z and with some manipulation at least we obtained the following simplified fourth order ODE [11]

$$\begin{aligned} &r^4 B_{1212} F^{(4)}(r) + 2r^3(r\, B'_{1212} + 3B_{1212})F^{(3)}(r) + r^2(r^2 B''_{1212} + 7\, r\, B'_{1212} \\ &+5B_{1212} + n^2(2B_{1122} + 2B_{2112} - B_{1111} - B_{2222}))F''(r) + r\,(r^2 B''_{1212} \\ &+r\, B'_{1212} - B_{1212} + n^2(2\, r\, B'_{2112} + 2\, r\, B'_{1122} - r\, B'_{1111} - r\, B'_{2222} + 2B_{1122} \\ &+2B_{2112} - B_{1111} - B_{2222}))\, F'(r) + (n^2 - 1)(r^2 B''_{1212} + r\, B'_{1212} - B_{1212} \\ &+n^2 B_{2121})\, F(r) = 0, \end{aligned} \tag{23}$$

and the relevant boundary conditions at $r = a, b$

$$\begin{aligned} &r^3 B_{1212}\, F'''(r) + r^2(r\, B'_{1212} + 4B_{1212})F''(r) + r(r\, B'_{1212} + (1 - n^2)B_{1212} \\ &+n^2(2B_{1122} + 2B_{2112} - B_{1111} - B_{2222}))F'(r) \\ &+(n^2 - 1)(r\, B'_{1212} + B_{1212})F(r) = 0, \\ &r^2\, F''(r) + r\, F'(r) + (n^2 - 1)F(r) = 0, \end{aligned} \tag{24}$$

where the prime denotes d/dr and n is the mode number. The fourth order ODE Eq. (23) and the boundary conditions Eq. (24) has been derived also by Haughton and Chen [36].

Cube

In the following section, a brief description of the derivations i.e. physical and the relevant governing equations of a cube has been chosen from Haughton [23] and Coman and Destrade [24].

Pure bending of a cube

The undeformed hyperelastic right cube occupies the region

$$\Omega = \{(X_1, X_2, X_3) \in \mathcal{R}^3 | -A \le X_1 \le A, -L \le X_2 \le L, 0 \le X_3 \le H\}, \tag{25}$$

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where A, L and H are the thickness, length and the height respectively. For this problem, we shall assume that all the variable and parameters which have dimension of length have been scaled by L. The cube is bent into a part of a right circular cylinder, such that the bent cube occupying the region

$$\Omega_t = \{(r,\theta,z) \in \mathcal{R} \times (0,2\pi] \times \mathcal{R} | a \le r \le b, -\omega_0 \le \theta \le \omega_0, 0 \le z \le H\,\lambda_3\}, \quad (26)$$

where a, b are the inner and outer radii, λ_3 is a prescribed axial stretch and ω_0 can serve as a control parameter. The deformation can now be described by

$$r = \sqrt{(\sqrt{1 + 4\omega_0^2 A^2} + 2X_1\omega_0)/(\omega_0^2\lambda_3)}, \theta = \omega_0 X_2, \quad z = \lambda_3 X_3. \quad (27)$$

Since the plate cannot be bent into itself, we required that $0 < \omega_0 \le \pi$. The angle of bending is $\varphi = 2\omega_0$. In view of the incompressible condition, the principal stretches are given by

$$\lambda_1 = \lambda^{-1}, \lambda_2 = \lambda(= \omega_0\, r), \lambda_3 = 1. \quad (28)$$

The two curved boundaries of the current configuration are taken to be traction-free.

Bifurcation analysis

Since most of the derivations of the cube and the cylinder are the same, we do not consider all such relations here and just write down the general fourth order differential equation. In view of relations Eqs. (18)-(22), the governing equations of the cube are obtained as follows

$$\begin{aligned}
&r^4 B_{1212} F^{(4)}(r) + 2r^3(r\,B'_{1212} + B_{1212})F'''(r) + r^2(r^2 B''_{1212} + r\,B'_{1212} \\
&-B_{1212} - m^2(B_{1111} + B_{2222} - 2B_{1122} - 2B_{2112}))F''(r) - r(r^2 B''_{1212} \\
&+r\,B'_{1212} - B_{1212} + m^2(r\,B'_{1111} + r\,B'_{2222} - 2\,r\,B'_{1122} - 2\,r\,B'_{2112} + 2B_{1122} \\
&+2B_{2112} - B_{1111} - B_{2222}))F'(r) + m^2(r^2 B''_{1212} + r\,B'_{1111} + r\,B'_{1212} \\
&+r\,B'_{2222} - 2\,r\,B'_{1122} - 2\,r\,B'_{2112} + 2B_{2112} + 2B_{1122} - B_{1111} - B_{2222} \\
&-B_{1212} - B_{2121} + m^2 B_{2121})F(r) = 0,
\end{aligned} \quad (29)$$

$$\begin{aligned}
&m^2(B_{1111} + B_{2222} + B_{1212} - 2B_{2112} - 2B_{1122})(F(r) - r\,F'(r)) \\
&+r^3 B_{1212}\,F'''(r) = 0, \\
&r^2\,F''(r) - r\,F'(r) + m^2\,F(r) = 0, (r = a, b)
\end{aligned} \quad (30)$$

where $m = \frac{\mu}{\omega_0}$ and $\mu = n\,\pi$. The inner and outer radii of the current configuration, respectively become

$$a, b = \frac{1}{\omega_0}\sqrt{\sqrt{1 + 4A^2\omega_0^2} \pm 2\,A\,\omega_0}. \quad (31)$$

Spherical shell

As will become clear later in this section first a brief description of the physical and also the relevant relations of the spherical shell subjected to an external hydrostatic pressure appears and later we continue the same procedure as used in the previous sections for an everted spherical shell. In the final part again we prefer to show what is necessary and omit the relations which are common with the cylinder. The general equilibrium equations and the boundary conditions for the mentioned spheres are given in the following section.

The buckling of a spherical shell

The undeformed spherical shell occupying the region,

$$0 < A \le R \le B, 0 \le \Theta \le \pi, 0 \le \Phi \le 2\pi, \quad (32)$$

in spherical polar coordinates (R, Θ, Φ), where A and B are the inner and outer radii. For the problems of this and the next sections, we shall assume that all the variable and parameters which have dimension of length have been scaled

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by B. The sphere is subjected to an external hydrostatic pressure so that the deformed spherical shell occupying the region

$$0 < a \leq r \leq b, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, \tag{33}$$

where (r, θ, ϕ) are spherical polar coordinates and a and b are the inner and outer radii in the deformed configuration. The deformation can now be described by

$$r = r(R), \theta = \Theta, \qquad \phi = \Phi, \tag{34}$$

where $r(R)$ is a smooth function. The principal stretches of the deformation is written by

$$\lambda_1 = r'(R), \lambda_2 = \lambda_3 = \lambda(= \frac{r}{R}). \tag{35}$$

With the use of the incompressible condition and by the aid of Eq. (35), we have

$$r^3 = R^3 - A^3 + a^3, \lambda_b^3 = 1 + A^3(\lambda_a^3 - 1), \tag{36}$$

where in view of Eq. (35), we get $\lambda_a = \frac{a}{A}$ and $\lambda_b = b$. Moreover, for the mentioned deformation, the only non-trivial equilibrium equation is

$$\frac{d\sigma_{11}}{dr} + \frac{2}{r}(\sigma_{11} - \sigma_{22}) = 0. \tag{37}$$

The principal Cauchy stresses and the boundary conditions are given respectively by Eq. (7) and Eq. (9).

The eversion of a spherical shell

The undeformed incompressible isotropic homogeneous elastic spherical shell is defined by Eq. (32). The deformation of this shell is given by

$$r = r(R), \theta = \pi - \Theta, \qquad \phi = \Phi, \tag{38}$$

where (r, θ, ϕ) are also spherical polar coordinates and $r(R)$ is a smooth, strictly decreasing function. The everted shell occupying the region as Eq. (33) and the principal stretches of the deformation are given by

$$\lambda_1 = -r'(R), \lambda_2 = \lambda_3 = \lambda(= \frac{r}{R}). \tag{39}$$

With the aid of the incompressible condition $\det(F) = 1$ and by using Eq.(39), we obtain

$$r^3 = 1 + a^3 - R^3, \lambda_b^3 = (1 + \lambda_a^3 - A^3)/A^3, \tag{40}$$

where $\lambda_a = a$ and $\lambda_b = \frac{b}{A}$. It is necessary to remind that for the mentioned deformation, the equilibrium equations Eq. (37) and the boundary conditions Eq. (14) and Eq. (15) are satisfied. The eversion problem is now fully defined. Indeed, with the use of the relations Eq. (37), Eq. (14) and Eq. (15) and in view of the explanations given by Haughton and Chen [36], the eversion conditions for Varga and Neo-Hookean materials are respectively

$$\frac{1}{\lambda_a^2} - \frac{1}{\lambda_b^2} + \frac{4}{\sqrt{3}}(\arctan(\frac{2\lambda_a - 1}{\sqrt{3}}) - \arctan(\frac{2\lambda_b - 1}{\sqrt{3}})) + \frac{2}{3}\ln(\frac{(\lambda_a + 1)^2(\lambda_b^2 - \lambda_b + 1)}{(\lambda_b + 1)^2(\lambda_a^2 - \lambda_a + 1)}) = 0, \tag{41}$$

$$\frac{1}{\lambda_b} - \frac{1}{\lambda_a} + \frac{1}{4}(\frac{1}{\lambda_a^4} - \frac{1}{\lambda_b^4}) = 0. \tag{42}$$

Bifurcation criterion

The incremental equations of non-linear elasticity are well documented, see Ogden [37] for example. For completeness we give a brief description. The incremental equilibrium equations for the spherical shell are defined by Eqs. (18)-(20). We consider the incremental displacement as $\dot{x} = (u(r, \theta, \phi), v(r, \theta, \phi), w(r, \theta, \phi))$. Although we shall study the eigen-value problem including the fourth order ODE,s, we consider the case where u, v and w are independent of ϕ. Then $\dot{x} = (u(r, \theta), v(r, \theta), 0)$ and the displacement gradient is defined by

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$$\Gamma = \begin{bmatrix} u_r & (u_\theta - v)/r & 0 \\ v_r & (u + v_\theta)/r & 0 \\ 0 & 0 & (u + v\cot\theta)/r \end{bmatrix}, \tag{43}$$

where subscripts denote partial derivatives. Since the material is incompressible, we have

$$tr\ \Gamma \equiv u_r + \frac{2u}{r} + \frac{v_\theta + v\cot\theta}{r} = 0. \tag{44}$$

With the use of relations Eqs. (18)-(20), Eq.(44) and in view of u, v and w which are independent of ϕ, the general governing equations are derived as follows

$$\begin{aligned} &r^4 B_{1212} F^{(4)}(r) + 2r^3(r\,B'_{1212} + 4B_{1212})F'''(r) + (r^2 B''_{1212} + 10\,r\,B'_{1212} \\ &+10B_{1212} + B_{2222} + B_{2121} - B_{2233} - B_{2112} + m\,(2B_{1122} + 2B_{2112} - B_{1111} \\ &-B_{2222}))r^2\,F''(r) + (2r^2 B''_{1212} + 4\,r\,B'_{1212} + r\,B'_{2222} + r\,B'_{2121} - r\,B'_{2233} \\ &-r\,B'_{2112} + 2B_{2222} + 2B_{2121} - 2B_{2233} - 2B_{2112} - 4B_{1212} + m\,(2\,r\,B'_{2112} \\ &+2\,r\,B'_{1122} - r\,B'_{1111} - r\,B'_{2222} + 4B_{2112} + 4B_{1122} - 2B_{1111} - 2B_{2222})) \\ &r\,F'(r) + (m-2)(r^2 B''_{1212} + 2\,r\,B'_{1212} + r\,B'_{2112} + r\,B'_{2233} - r\,B'_{2121} \\ &-r\,B'_{2222} + B_{2222} - 2B_{1212} - B_{2112} - B_{2233} + (m+1)B_{2121})F(r) = 0, \end{aligned} \tag{45}$$

and the relevant boundary conditions at $r = a, b$ are

$$\begin{aligned} &r^3 B_{1212}\,F'''(r) + 4r^2 B_{1212}\,F''(r) + (B_{2121} + B_{2222} - B_{2112} - B_{2233} \\ &+m\,(2B_{2112} + 2B_{1122} - B_{1111} - B_{2222} - B_{1212}))r\,F'(r) + (m-2)(B_{2112} \\ &+B_{2233} - B_{2121} - B_{2222})F(r) = 0, \\ &r^2\,F''(r) + 2\,r\,F'(r) + (m-2)F(r) = 0, \end{aligned} \tag{46}$$

where $m = n\,(n+1)$ and the prime denotes d/dr.

Asymptotic results

For all the problems of this article and for large n limit, we look for a WKB solution of the following form as described in Fu and Sanjaranipour [20] and Sanjaranipour [7]

$$F(r) = \left(F_0(r) + \tfrac{1}{n}F_1(r) + \tfrac{1}{n^2}F_2(r) + \tfrac{1}{n^3}F_3(r) + \cdots\right)exp(n\textstyle\int S(r)dr), \tag{47}$$

where $F_0(r), F_1(r), F_2(r), \ldots$ and S are functions of r and are to be determined. As we mentioned previously, it seems that the roots of $S(r)$ for Neo-Hookean material are not repeated while for Varga material are repeated and also the differential equations of $F_0(r)$, $F_1(r)$, $F_2(r)$, ... of the WKB expansion are first and second order respectively for Neo-Hookean and Varga materials.

WKB analysis of the buckling of a Neo-Hookean cylindrical shell

For the solution of this problem, the following standard form of WKB expansion (see, e.g., Bender and Orszag [30])) has been used in Sanjaranipour [8]

$$F(r) = exp\left(n\int_a^r S(r)dr\right), S(r) = S_0(r) + \frac{S_1(r)}{n} + \frac{S_2(r)}{n^2} + \ldots = \sum_{m=0}^{\infty}\frac{S_m(r)}{n^m}. \tag{48}$$

It should be emphasized that for the aims of this article which is comparing the differential equations of $F_0(r), F_1(r), F_2(r), \ldots$ for different scenarios, we use the particular and equivalent form of WKB expansion i.e. Eq. (47) instead of Eq. (48). To determine $S(r)$, we substitute Eq. (47) into Eq. (23) and equate the coefficients of the like powers of n, we obtain an infinite number of differential equations for $(r), F_1(r), F_2(r), \ldots$. To leading order the resulting four un-repeated roots of $S(r)$ are

$$S^{(1)}(r) = \frac{1}{r}, S^{(2)}(r) = \frac{-1}{r}, S^{(3)}(r) = \frac{r}{q+r^2}, S^{(4)}(r) = \frac{-r}{q+r^2}, \tag{49}$$

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where $q = A^2 - a^2$. By equating the coefficients of the next order terms and in view of $S(r)$ from Eq. (49) and with some manipulation at least, we obtain the following two first order differential equations for $F_0(r)$,

$$\begin{aligned}(F_0^{(i)})' - \frac{q^2 + 2qr^2 + 2r^4}{r(q+r^2)(q+2r^2)} F_0^{(i)} &= 0, (i = 1,2)\\ (F_0^{(i)})' - \frac{2r^3}{(q+r^2)(q+2r^2)} F_0^{(i)} &= 0, (i = 3,4)\end{aligned} \tag{50}$$

and the solutions are

$$F_0^{(1)}(r) = F_0^{(2)}(r) = \frac{r\sqrt{q + r^2}}{\sqrt{q + 2r^2}}, F_0^{(3)}(r) = F_0^{(4)}(r) = \frac{q + r^2}{\sqrt{q + 2r^2}}, \dots \tag{51}$$

where superscripts here correspond to those in Eq. (49). To find the differential equations satisfied by $F_1(r)$, we have to continue the analysis to the next order. With the use of $S^{(i)}(r)$ and $F_0^{(i)}(r)$ from Eq. (49) and Eq. (51), the four non-homogeneous first order differential equations of $F_1^{(i)}(r)$ are obtained as

$$\begin{aligned}(F_1^{(i)})' - \frac{q^2 + 2qr^2 + 2r^4}{r(q+r^2)(q+2r^2)} F_1^{(i)} &= \frac{(-1)^{i+1} q r^2 (8q^3 + 15q^2r^2 - 2qr^4 - 12r^6)}{2(q + r^2)^{3/2}(q + 2r^2)^{7/2}}, \quad (i = 1,2)\\ (F_1^{(i)})' - \frac{2r^3}{(q+r^2)(q+2r^2)} F_1^{(i)} &= \frac{(-1)^{i+1} q (q + r^2)(3q^3 + 17q^2r^2 + 34qr^4 + 12r^6)}{2r^3(q + 2r^2)^{7/2}}, (i = 3,4)\end{aligned} \tag{52}$$

and the particular integrals of Eq. (52) are

$$\begin{aligned}F_1^{(i)}(r) &= \frac{(-1)^i r\sqrt{q + r^2}}{8\sqrt{q + 2r^2}} \{\frac{q(q^2 - 9qr^2 - 16r^4)}{(q + r^2)(q + 2r^2)^2} + 2\ln(\frac{q + r^2}{q + 2r^2})\}, (i = 1,2)\\ F_1^{(i)}(r) &= \frac{(-1)^i (q + r^2)}{8\sqrt{q + 2r^2}} \{\frac{q(6q^2 + 23qr^2 + 16r^4)}{r^2(q + 2r^2)^2} + 2\ln(\frac{r^2}{q + 2r^2})\}. \ (i = 3,4)\end{aligned} \tag{53}$$

Now we may write down the general solution as

$$F(r) = \sum_{i=1}^{4} c_i F^{(i)}(r) E^{(i)}(r), \tag{54}$$

where $c_i (i = 1,2,3,4)$ are disposable constants and we have

$$F^{(i)}(r) = F_0^{(i)}(r) + \frac{F_1^{(i)}(r)}{n} + \frac{F_2^{(i)}(r)}{n^2} + \cdots, E^{(i)}(r) = \exp\left(n\int_a^r S^{(i)}(x)dx\right). \tag{55}$$

On substituting Eq. (54) into the boundary conditions Eq. (24), we obtain

$$\sum_{i=1}^{4} c_i \{\alpha^{(i)}(r), \gamma^{(i)}(r)\}^T E^{(i)}(r) = 0, (r = a, b) \tag{56}$$

Where

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$$\alpha^{(i)}(r) = (1 + r^2(S^{(i)})^2)F_0^{(i)} + \frac{1}{n}((1 + r^2(S^{(i)})^2)F_1^{(i)} + (r\,S^{(i)} + r^2(S^{(i)})')F_0^{(i)}$$
$$+2r^2S^{(i)}(F_0^{(i)})') + \frac{1}{n^2}(-F_0^{(i)} + (r\,S^{(i)} + r^2(S^{(i)})')F_1^{(i)} + (1 + r^2(S^{(i)})^2)$$
$$F_2^{(i)} + r\,(F_0^{(i)})' + 2r^2S^{(i)}(F_1^{(i)})' + r^2(F_0^{(i)})'') + O(\frac{1}{n^3}),$$
$$\gamma^{(i)}(r) = \frac{S^{(i)}}{r\,(q + r^2)}(2q^2 + 4\,q\,r^2 + 3r^4 - r^2(q + r^2)^2(S^{(i)})^2)F_0^{(i)} + \frac{1}{n\,r^2(q + r^2)} \tag{57}$$
$$((q + r^2)F_0^{(i)}(q - r^2 - r^2S^{(i)}((2\,q + 4r^2)S^{(i)} + 3\,r\,(q + r^2)(S^{(i)})'))$$
$$+r(2q^2 + 4\,q\,r^2 + 3r^4 - r^2(q + r^2)^2(S^{(i)})^2)S^{(i)}F_1^{(i)} + r\,(2q^2 + 4\,q\,r^2$$
$$+3r^4 - 3r^2(q + r^2)^2(S^{(i)})^2)(F_0^{(i)})') + O(\frac{1}{n^2}).$$

We may write the boundary conditions as a matrix equation of the form $CM = 0$, where $C = [c_1, c_2, c_3, c_4]$ and

$$M = (M_{ij}) = \begin{bmatrix} \alpha^{(1)}(a) & \alpha^{(2)}(a) & \alpha^{(3)}(a) & \alpha^{(4)}(a) \\ \gamma^{(1)}(a) & \gamma^{(2)}(a) & \gamma^{(3)}(a) & \gamma^{(4)}(a) \\ E_1\alpha^{(1)}(b) & E_2\alpha^{(2)}(b) & E_3\alpha^{(3)}(b) & E_4\alpha^{(4)}(b) \\ E_1\gamma^{(1)}(b) & E_2\gamma^{(2)}(b) & E_3\gamma^{(3)}(b) & E_4\gamma^{(4)}(b) \end{bmatrix}. \tag{58}$$

In order to obtain the non-trivial solution, a certain 4×4 determinant has to vanishes i.e. $\det(M) = 0$.

Asymptotic result for $A - 1 = O(1)$

In the case of $A - 1 = O(1)$, $E^{(1)}(b) = E^{(3)}(b)$ are exponentially large whereas $E^{(2)}(b) = E^{(4)}(b)$ are exponentially small; we may decouple the equations of $\det(M) = 0$ into two pairs of equations

$$\begin{bmatrix} \alpha^{(1)}(b) & \alpha^{(3)}(b) \\ \gamma^{(1)}(b) & \gamma^{(3)}(b) \end{bmatrix}\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = 0, \quad \text{and} \quad \begin{bmatrix} \alpha^{(2)}(a) & \alpha^{(4)}(a) \\ \gamma^{(2)}(a) & \gamma^{(4)}(a) \end{bmatrix}\begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = 0. \tag{59}$$

By considering $\{\hat{\alpha}^{(i)}(r), \hat{\gamma}^{(i)}(r)\} = \{\alpha^{(i)}(r), \gamma^{(i)}(r)\}/F^{(i)}(r)$ (in order to eliminate $F^{(2)}(r)$ from our derivations), $(59)_{1,2}$ can be replaced respectively by

$$\begin{bmatrix} \hat{\alpha}^{(1)}(b) & \hat{\alpha}^{(3)}(b) \\ \hat{\gamma}^{(1)}(b) & \hat{\gamma}^{(3)}(b) \end{bmatrix}\begin{bmatrix} c_1F^{(1)}(b) \\ c_3F^{(3)}(b) \end{bmatrix} = 0, \quad \text{or} \begin{bmatrix} \hat{\alpha}^{(2)}(a) & \hat{\alpha}^{(4)}(a) \\ \hat{\gamma}^{(2)}(a) & \hat{\gamma}^{(4)}(a) \end{bmatrix}\begin{bmatrix} c_2F^{(2)}(b) \\ c_4F^{(4)}(b) \end{bmatrix} = 0. \tag{60}$$

To obtain a non-trivial solution, we must have

$$\begin{vmatrix} \hat{\alpha}^{(1)}(b) & \hat{\alpha}^{(3)}(b) \\ \hat{\gamma}^{(1)}(b) & \hat{\gamma}^{(3)}(b) \end{vmatrix} = 0, \text{ or } \begin{vmatrix} \hat{\alpha}^{(2)}(a) & \hat{\alpha}^{(4)}(a) \\ \hat{\gamma}^{(2)}(a) & \hat{\gamma}^{(4)}(a) \end{vmatrix} = 0. \tag{61}$$

By expanding Eq. (61) (the details are given in Sanjaranipour [8]), we obtain

$$(1 - 3\lambda_a^2 - \lambda_a^4 - \lambda_a^6) + (1 + 2\lambda_a^2 - 2\lambda_a^4 + 2\lambda_a^6 + \lambda_a^8)\frac{1}{n} - \frac{1}{2\lambda_a^2(1 + \lambda_a^2)^2}(12 + 20$$
$$\lambda_a^2 - 12\lambda_a^4 - 43\lambda_a^6 - 65\lambda_a^8 - 10\lambda_a^{10} + 6\lambda_a^{12} + 9\lambda_a^{14} + 3\lambda_a^{16})\frac{1}{n^2} + O(\frac{1}{n^3}) = 0, \tag{62}$$

where $\lambda_a = \frac{a}{A}$. By expanding λ_a in term of $\frac{1}{n}$, we obtain from E. (62),

$$\lambda_a = 0.543689 + 0.352202/n - 3.713199/n^2 + \cdots. \tag{63}$$

This expression is independent of A and as expected is exactly similar to the obtained expansion in Sanjaranipour [8].

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Asymptotic results for $A - 1 = O(\frac{1}{n})$

Since for small $A-1$, $b-a$ is small, the exponentials E_1 and E_3 are no longer exponentially large and Eq.(59) fails to approximate $\det(M) = 0$. In the boundary layer we expand A as

$$A = 1 + \frac{1}{n}\xi, \tag{64}$$

where ξ is an $O(1)$ constant, and we look for an asymptotic solution for λ_a of the form

$$\lambda_a = \eta_1 + \frac{1}{n}\eta_2 + \cdots, \tag{65}$$

where η_1 and η_2 are to be determined. On substituting Eq. (64) and Eq. (65) into relations $a = \lambda_a A$, $b = \lambda_b$ and by using Eq. (5), we obtain

$$a = \eta_1 + (\xi\eta_1 + \eta_2)\frac{1}{n} + \cdots, \qquad b = \eta_1 + \frac{-\xi + \xi\eta_1^2 + \eta_1\eta_2}{\eta_1}\frac{1}{n} + \cdots . \tag{66}$$

By substituting a and b from Eq. (66) into $M = (M_{ij})$ and by expanding $\det(M)$, with the aid of Mathematica, to the leading order $\det(M) = 0$ gives

$$16\,z\,(z^2+1)^2 - (1 + 4\,z + 2z^2 + z^4)^2\cosh(\xi - \frac{\xi}{z}) + (1 - 4\,z + 2z^2 + z^4)^2\cosh(\xi + \frac{\xi}{z}) = 0, \tag{67}$$

where $z = \eta_1^2$. The equation satisfied by η_2 is obtained by collecting the coefficients of the next order term and is given by

$$\begin{aligned}
&(2\xi^2\cosh(\tfrac{\xi}{\eta_1^2})\sinh(\xi) + \eta_1(8\eta_1^5(8\xi(1+\eta_1^4)(1-\eta_1^2+\eta_1^4+\eta_1^6) - \eta_1(8\\
&+\cosh(\xi - \tfrac{\xi}{\eta_1^2}) - 9\xi\sinh(\xi - \tfrac{\xi}{\eta_1^2}) + 8\eta_1^2(2+2\eta_1^4+\eta_1^6)\eta_2) + \cosh(\tfrac{\xi}{\eta_1^2})(8\\
&\cosh(\xi)\eta_1^5(-8\xi(1+\eta_1^4)(1-\eta_1^2+\eta_1^4+\eta_1^6) - \eta_1(-9+8\eta_1^2(2+2\eta_1^4+\eta_1^6))\\
&\eta_2) + \xi\sinh(\xi)(-\xi(\eta_1 - 37\eta_1^3 + 12\eta_1^5 + 40\eta_1^7 - 10\eta_1^9 - 6\eta_1^{11} - 4\eta_1^{13} + 2\eta_1^{15}\\
&+\eta_1^{17} + 3\eta_1^{19}) - 4\eta_2 - 4\eta_1^2(1+\eta_1^2(20+\eta_1^2(2+\eta_1^2(1+\eta_1^2)(6+4\eta_1^4+\eta_1^8))))\\
&\eta_2)) + \sinh(\tfrac{\xi}{\eta_1^2})\eta_1(\xi\cosh(\xi)(-\xi(16 - 7\eta_1^2 + 9\eta_1^4 + 4\eta_1^6 - 12\eta_1^8 - 2\eta_1^{10} - 18\\
&\eta_1^{12} + 4\eta_1^{14} + 4\eta_1^{16} + \eta_1^{18} + \eta_1^{20}) + 8\eta_1(4 + 4\eta_1^2 - \eta_1^4 + 8\eta_1^6 + 4\eta_1^8 + 4\eta_1^{10})\eta_2)\\
&+2\sinh(\xi)(\xi + \eta_1(\xi\eta_1(4 - 21\eta_1^2 + 80\eta_1^4 + 2\eta_1^6 + 24\eta_1^8 - 2\eta_1^{10} + 16\eta_1^{12}\\
&+13\eta_1^{14} + 4\eta_1^{16} + 7\eta_1^{18}) - 2(1 + 3\eta_1^2 - 18\eta_1^4 + 20\eta_1^6 - 18\eta_1^8 - 6\eta_1^{10}\\
&-20\eta_1^{12} - 12\eta_1^{14} - 7\eta_1^{16} - 5\eta_1^{18})\eta_2))))))/(\eta_1^4+\eta_1^6) = 0.
\end{aligned} \tag{68}$$

For any given mode number n and any value of A close to unity, the corresponding value of ξ is determined by Eq. (64) and solving Eq. (67) and Eq. (68) gives the corresponding values of η_1 and η_2. The value of λ_a is then calculated according to Eq. (65). In Fig. 1, we have shown the comparison between the numerical and asymptotic results of the outer and inner layers for n=8, 10, 15, 20.

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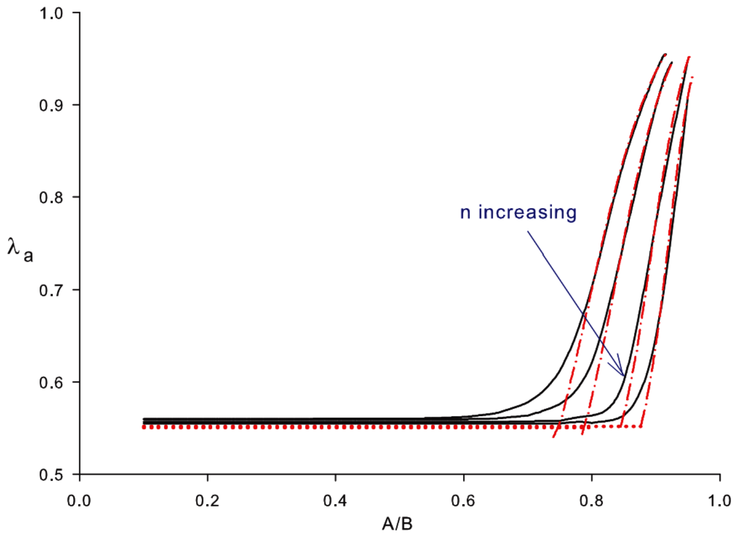

Fig. 1. Bifurcation curves of Neo-Hookean cylindrical shell subjected to an external hydrostatic pressure, for n=8, 10, 15, 20. Solid lines: numerical results; dotted lines: outer layer results of WKB; dashed dotted lines represent results of inner layer of WKB.

Indeed the asymptotic results approximate the numerical results extremely well over the region of their validity and this is true since WKB approximation give us better and precise results for high mode numbers. In view of the explanations given in this section and as we expected, Fig. 1 is similar to Fig. 4 in Sanjaranipour [8]. It should be remind that the same expansion i.e. Eq. (47) was used by Sanjaranipour [7] for the problem of circular cylindrical tube made of Varga material and the obtained expressions for $S(r)$, $F_0(r)$ and $F_1(r)$ are

$$S^{(1)}(r) = S^{(3)}(r) = \frac{1}{\sqrt{k+r^2}}, S^{(2)}(r) = S^{(4)}(r) = \frac{-1}{\sqrt{k+r^2}}, \tag{69}$$

$$(F_0^{(i)})'' + (\frac{2}{r} + \frac{r}{k+r^2} - \frac{4\,r}{k+2r^2})(F_0^{(i)})' + (\frac{1}{k+r^2} - \frac{4}{k+2r^2})F_0^{(i)} = 0, (i = 1,2,3,4) \tag{70}$$

where $k = A^2(1-\lambda_a^2)$ and the solutions of Eq. (70) are

$$F_0^{(1)}(r) = F_0^{(2)}(r) = \frac{1}{r}, \qquad F_0^{(3)}(r) = F_0^{(4)}(r) = \sqrt{k+r^2}. \tag{71}$$

In view of Eq. (69) and by considering Eq. (71) the differential equations of $F_1(r)$ are respectively

$$\left(F_1^{(i)}\right)'' + \left(\frac{2}{r} + \frac{r}{k+r^2} - \frac{4\,r}{k+2r^2}\right)\left(F_1^{(i)}\right)' + \left(\frac{1}{k+r^2} - \frac{4}{k+2r^2}\right)F_1^{(i)} = \begin{cases} 0 & i = 1,2 \\ \dfrac{(-1)^i 2k^2}{r\,(k+r^2)(k+2r^2)} & i = 3,4 \end{cases} \tag{72}$$

and the relevant solutions are

$$F_1^{(1)}(r) = F_1^{(2)}(r) = 0, \quad F_1^{(3)}(r) = -\frac{k+2r^2}{2r}, \quad F_1^{(4)}(r) = \frac{k+2r^2}{2r}. \tag{73}$$

As we suggested, it is now confirmed that for the two mentioned problems the roots of $S(r)$ are u-repeated for Neo-Hookean, while are repeated for Varga materials and also the differential equations of $F_0(r)$, $F_1(r)$, $F_2(r)$, ... are first and second order respectively for the mentioned materials.

WKB analysis of the buckling of a Neo-Hookean everted cylinder

It is necessary to note that this problem has been solved by Fu and Lin [21]. They used the same expansion as we used in this article but since the differential equations of $F_0(r)$ and $F_1(r)$ are not appeared on their paper, we have done the analysis once again and a brief description of the obtained results are given here. To the leading order four un-repeated roots of $S(r)$ are

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$$S^{(1)}(r)=\frac{1}{r},S^{(2)}(r)=\frac{-1}{r},S^{(3)}(r)=\frac{r\,\lambda_z}{q-r^2\lambda_z},S^{(4)}(r)=\frac{-r\,\lambda_z}{q-r^2\lambda_z}, \tag{74}$$

where $q=1+\lambda_z a^2$ and the first order differential equations of $F_0(r)$ are obtained as

$$\begin{aligned}&(F_0^{(i)})'-\frac{q^2-2\,q\,r^2\lambda_z+2r^4\lambda_z^2}{r(q-2r^2\lambda_z)(q-r^2\lambda_z)}F_0^{(i)}=0,(i=1,2)\\&(F_0^{(i)})'-\frac{2r^3\lambda_z^2}{(q-2r^2\lambda_z)(q-r^2\lambda_z)}F_0^{(i)}=0,(i=3,4)\end{aligned} \tag{75}$$

where superscripts here correspond to those in Eq. (74). The solutions of $F_0(r)$ are

$$F_0^{(1)}(r)=F_0^{(2)}(r)=\frac{r\sqrt{q-r^2\lambda_z}}{\sqrt{q-2r^2\lambda_z}},F_0^{(3)}(r)=F_0^{(4)}(r)=\frac{q-r^2\lambda_z}{\sqrt{q-2r^2\lambda_z}}. \tag{76}$$

The differential equations satisfied by $F_1(r)$, are the following four first order inhomogeneous differential equations

$$\begin{aligned}&\left(F_1^{(i)}\right)'-\frac{q^2-2\,q\,r^2\lambda_z+2r^4\lambda_z^2}{r(q-2r^2\lambda_z)(q-r^2\lambda_z)}F_1^{(i)}=\frac{(-1)^i\,q\,r^2\lambda_z(8q^3-15q^2r^2\lambda_z-2\,q\,r^4\lambda_z^2+12r^6\lambda_z^3)}{2(q-2r^2\lambda_z)^{\frac{7}{2}}(q-r^2\lambda_z)^{\frac{3}{2}}},\\&\quad(i=1,2)\\&\left(F_1^{(i)}\right)'-\frac{2r^3\lambda_z^2}{(q-2r^2\lambda_z)(q-r^2\lambda_z)}F_1^{(i)}=\frac{(-1)^{i+1}\,q(q-r^2\lambda_z)(3q^3-17q^2r^2\lambda_z+34\,q\,r^4\lambda_z^2-12r^6\lambda_z^3)}{2r^3\lambda_z(q-2r^2\lambda_z)^{\frac{7}{2}}}.\\&\quad(i=3,4)\end{aligned} \tag{77}$$

The particular integrals of Eq. (77), which can be found easily, are

$$\begin{aligned}&F_1^{(i)}(r)=\frac{(-1)^i\,r\sqrt{q-r^2\lambda_z)}}{4\sqrt{q-2r^2\lambda_z}}\{\frac{q(q^2+9\,q\,r^2\lambda_z-16r^4\lambda_z^2)}{2(q-2r^2\lambda_z)^2(q-r^2\lambda_z)}+\ln(\frac{q-r^2\lambda_z}{q-2r^2\lambda_z})\},(i=1,2)\\&F_1^{(i)}(r)=\frac{(-1)^i(q-r^2\lambda_z)}{4\sqrt{q-2r^2\lambda_z}}\{\frac{q(6q^2-23\,q\,r^2\lambda_z+16r^4\lambda_z^2)}{2r^2\lambda_z(q-2r^2\lambda_z)^2}+\ln(\frac{q-2r^2\lambda_z}{r^2})\}.(i=3,4)\end{aligned} \tag{78}$$

By expanding Eq. (61) with the use of Eq. (74), Eq. (76) and Eq. (78), yields

$$\begin{aligned}&(1-4a^2\lambda_z+2a^4\lambda_z^2+a^8\lambda_z^4)+\frac{1}{n}(1+3a^2\lambda_z+3a^8\lambda_z^4+a^{10}\lambda_z^5+a^8\lambda_z^4)\\&-\frac{1}{2a^2\lambda_z n^2(1-a^2\lambda_z)^3}(12-20a^2\lambda_z-8a^4\lambda_z^2+49a^6\lambda_z^3-131a^8\lambda_z^4-39a^{10}\lambda_z^5\\&-31a^{12}\lambda_z^6-21a^{14}\lambda_z^7-5a^{16}\lambda_z^8-a^{18}\lambda_z^9+3a^{20}\lambda_z^{10})+O(\frac{1}{n^3})=0.\end{aligned} \tag{79}$$

$$\begin{aligned}&A=0.387252+1.223959/n-16.048113/n^2+\cdots,\\&\lambda_z=1.139181-0.689422/n+10.277128/n^2+\cdots,\\&a=0.509394+0.761078/n-9.772518/n^2+\cdots,\\&b=1.002828+0.196853/n-3.381406/n^2+\cdots.\end{aligned} \tag{80}$$

Table1. The critical values of $\frac{A}{B}$ and the corresponding values of $\lambda_z(=\frac{1}{L})$ and $\frac{b}{B}$ for various mode numbers and related to Neo-Hookean everted cylinder.

	CMM			WKB		
n	$\frac{A}{B}$	λ_z	$\frac{b}{B}$	$\frac{A}{B}$	λ_z	$\frac{b}{B}$

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7	0.406912	1.128420	1.005787	0.234590	1.250430	0.961942
8	0.412655	1.125392	1.006575	0.289495	1.213580	0.974600
9	0.416873	1.123200	1.007134	0.325122	1.189460	0.982955
10	0.419736	1.121727	1.007502	0.349166	1.173010	0.988699
11	0.421546	1.120802	1.007731	0.365892	1.161440	0.992778
12	0.422585	1.120274	1.007861	0.377803	1.153100	0.995751
13	0.423075	1.120025	1.007922	0.386443	1.146960	0.997962
14	0.423184	1.119970	1.007936	0.392799	1.142370	0.999637
15	0.423028	1.120049	1.007916	0.397524	1.138900	1.000920
16	0.422692	1.120219	1.007874	0.401061	1.136240	1.001920
17	0.422234	1.120452	1.007817	0.403719	1.134190	1.002710
18	0.421694	1.120727	1.007750	0.405718	1.132600	1.003330
19	0.421102	1.121029	1.007675	0.407216	1.131360	1.003820
20	0.420479	1.121347	1.007597	0.408329	1.130400	1.004220
30	0.414360	1.124502	1.006803	0.410219	1.127620	1.005630
40	0.409712	1.126937	1.006175	0.407820	1.128370	1.005640
50	0.406331	1.128729	1.005705	0.405311	1.129500	1.005410
60	0.403805	1.13008	1.005346	0.403193	1.13055	1.00517
70	0.401857	1.131128	1.005064	0.401462	1.13143	1.00495
80	0.400314	1.131963	1.004839	0.400044	1.13217	1.00476

As we expected these expansions are exactly the same as those of Fu and Lin [21]. For the special case $n = 20$, we obtained $a = 0.523017$ and $b = 1.00422$ and since the roots of $(1 + \lambda_z a^2 - \lambda_z r^2)$, $(1 + \lambda_z a^2 - 2\lambda_z r^2)$ are $r = 1.076191$ and $r = 0.760982$, respectively, hence $r = 0.760982$ is the turning point. According to what Fu and Lin [21] suggested, and due to our results in Table 1, the difference between the results of compound matrix and WKB methods is the existence of the turning point. The obtained numerical results in Table 1 are similar to that of Haughton and Orr [19]. (Note: CMM in Table 1 is used for compound matrix method.)

WKB analysis of the buckling of a Varga everted cylinder

This eigen-value problem previously has been solved by Fu and Sanjaranipour [20] but as a system of equations. Since our goal in this article is to analysis and compare the differential equations of $F_0(r)$, $F_1(r)$, $F_2(r)$, ..., we are solving here the fourth order differential equation instead of a system of equations. We use the same procedure as we have done in the previous section and just write down what is necessary. To the leading order two repeated roots of $S(r)$ are

$$S^{(1)}(r) = S^{(3)}(r) = \frac{\sqrt{\lambda_z}}{\sqrt{q - r^2\lambda_z}}, S^{(2)}(r) = S^{(4)}(r) = \frac{-\sqrt{\lambda_z}}{\sqrt{q - r^2\lambda_z}}, \tag{81}$$

where $q = 1 + \lambda_z a^2$. In view of $S(r)$ from Eq. (81), the next order term is automatically satisfied (which happens when the roots of $S(r)$ are repeated (see, e.g., Fu and Sanjaranipour [20]). In order to obtain the differential equation of $F_0(r)$, we have to continue our analysis to the next order. In view of Eq. (81), the second order differential equation of $F_0(r)$ will be

$$(F_0^{(i)})'' + \frac{2\,q - 3r^2\lambda_z}{r(q - r^2\lambda_z)}(F_0^{(i)})' = 0, (i = 1,2,3,4) \tag{82}$$

and two independent solutions are

$$F_0^{(1)}(r) = F_0^{(2)}(r) = 1, F_0^{(3)}(r) = F_0^{(4)}(r) = \frac{\sqrt{q\, -\, r^2\lambda_z}}{q\, r}, \tag{83}$$

where superscripts here correspond to those in Eq. (81). In a similar way, we obtain a second order non-homogenous ODE for $F_1(r)$ as follows

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$$(F_1^{(i)})'' + \frac{2\,q - 3r^2\lambda_z}{r(q - r^2\lambda_z)}(F_1^{(i)})' = \begin{cases}(-1)^i\sqrt{\lambda_z}/(r\sqrt{q - r^2\lambda_z}), & i = 1,2\\ (-1)^{i+1}\lambda_z^{3/2}/(q^2 - q\,r^2\lambda_z). & i = 3,4\end{cases} \tag{84}$$

The particular integrals of Eq. (84) are

$$F_1^{(i)}(r) = \begin{cases}\frac{(-1)^i}{2}\arctan(\frac{r\sqrt{\lambda_z}}{\sqrt{q - r^2\lambda_z}}), i = 1,2\\ \frac{(-1)^{i+1}}{2\,q\,r}(r\sqrt{\lambda_z} - \sqrt{q - r^2\lambda_z}\arctan(\frac{r\sqrt{\lambda_z}}{\sqrt{q - r^2\lambda_z}})). \quad i = 3,4\end{cases} \tag{85}$$

In a similar way and after lengthy calculations, we obtain

$$F_2^{(i)}(r) = \frac{1}{8}\arctan^2(\frac{r\sqrt{\lambda_z}}{\sqrt{q - r^2\lambda_z}}), \qquad (i = 1, 2)$$
$$F_2^{(i)}(r) = \frac{-1}{8\,q\,r}\arctan(\frac{r\sqrt{\lambda_z}}{\sqrt{q - r^2\lambda_z}})(2\,r\sqrt{\lambda_z} - \sqrt{q - r^2\lambda_z}\arctan(\frac{r\sqrt{\lambda_z}}{\sqrt{q - r^2\lambda_z}})). \qquad (i = 3, 4) \tag{86}$$

Expanding Eq. (61) with the use of Eq. (81), Eq. (83), Eq. (85) and Eq. (86), yields

$$(1 - 3a^2\lambda_z) - \frac{1}{2}\,a\sqrt{\lambda_z}(3 - a^2\lambda_z)\frac{1}{n} - \frac{1}{4}(4 - 9a^2\lambda_z - a^4\lambda_z^2)\frac{1}{n^2} + O(\frac{1}{n^3}) = 0. \tag{87}$$

$$\begin{aligned} A &= 0.430545 - 0.467349/n - 0.269644/n^2 + \cdots,\\ \lambda_z &= 1.058227 + 0.114162/n + 0.136329/n^2 + \cdots,\\ a &= 0.561242 - 0.246296/n - 0.063623/n^2 + \cdots,\\ b &= 1.041537 + 0.009974/n - 0.061965/n^2 + \cdots. \end{aligned} \tag{88}$$

These expansions are exactly similar to (but one term more than) those obtained by Fu and Sanjaranipour [20]. The numerical results we offered on Table 2 are the same as those of Haughton and Orr [19]. It is seen from this table that, there is a good agreement between the WKB and the corresponding numerical results over almost the whole mode number regime.

Table 2. The critical values of $\frac{A}{B}$ and the corresponding values of $\lambda_z(=\frac{1}{L})$ and $\frac{b}{B}$ for various mode numbers and related to Varga everted cylinder.

	CMM			WKB		
n	$\frac{A}{B}$	λ_z	$\frac{b}{B}$	$\frac{A}{B}$	λ_z	$\frac{b}{B}$
5	0.335991	1.084341	1.040559	0.326290	1.086510	1.041050
10	0.380802	1.071195	1.041798	0.381114	1.071010	1.041920
15	0.398089	1.066500	1.041894	0.398190	1.066440	1.041930
20	0.406462	1.064299	1.041867	0.406504	1.064280	1.041880
30	0.414655	1.062190	1.041797	0.414668	1.062180	1.041800
40	0.418688	1.061169	1.041746	0.418693	1.061170	1.041750
50	0.421088	1.060566	1.041711	0.421091	1.060560	1.041710
60	0.422680	1.060168	1.041686	0.422681	1.060170	1.041690
70	0.423813	1.059886	1.041667	0.423814	1.059890	1.041670
		1.059675	1.041652	0.424661	1.059670	

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80	0.424661	1.059512	1.041640	0.425319	1.059510	1.041650
90	0.425319	1.059382	1.041631	0.425845	1.059380	1.041640
100	0.425845	1.058994	1.041601	0.427418	1.058990	1.041630
150	0.427418	1.058801	1.041586	0.428202	1.058800	1.041600
200	0.428202	1.058685	1.041576	0.428672	1.058690	1.041590
250	0.428672	1.058609	1.041570	0.428985	1.058610	1.041580
300	0.428985	1.058554	1.041565	0.429208	1.058550	1.041570
350	0.429208	1.058513	1.041562	0.429375	1.058510	1.041570
400	0.429375					1.041560

WKB analysis of pure bending of a Neo-Hookean cube

It should be emphasized that this problem has been analyzed by Coman and Destrade [24]. The same WKB expansion was used in their article as we applied here but since the differential equations of $F_0(r)$ and $F_1(r)$ are not expressed, we not only re-derived the relevant equations of $F_0(r)$ and $F_1(r)$ but also extended our analysis for different $\frac{A}{L}$'s instead of $\frac{A}{L}=1$. It is also worthwhile noting that from now on and up to the end of the next section n will be replaced by μ. By continuing the same procedure as the previous sections, we obtain four un-repeated roots of $S(r)$ as

$$S^{(1)}(r)=\frac{1}{r\,\omega_0},S^{(2)}(r)=\frac{-1}{r\,\omega_0},S^{(3)}(r)=r\,\omega_0,S^{(4)}(r)=-r\,\omega_0, \tag{89}$$

and we have the following two first order differential equations of $F_0(r)$

$$\begin{aligned}(F_0^{(i)})'-\frac{2}{r-r^5\omega_0^4}F_0^{(i)}&=0,(i=1,2)\\(F_0^{(i)})'-\frac{1+r^4\omega_0^4}{r-r^5\omega_0^4}F_0^{(i)}&=0,(i=3,4)\end{aligned} \tag{90}$$

where the relevant solutions are

$$F_0^{(1)}(r)=F_0^{(2)}(r)=\frac{r^2}{\sqrt{1-r^4\omega_0^4}},F_0^{(3)}(r)=F_0^{(4)}(r)=\frac{r}{\sqrt{1-r^4\omega_0^4}}. \tag{91}$$

Continuing in this manner, we obtain the followings differential equations satisfied by $F_1(r)$

$$\begin{aligned}(F_1^{(i)})'-\frac{2}{r-r^5\omega_0^4}F_1^{(i)}&=\frac{(-1)^i6r^5\omega_0^5(3+r^4\omega_0^4)}{(1-r^4\omega_0^4)^{7/2}}, \qquad (i=1,2)\\(F_1^{(i)})'-\frac{1+r^4\omega_0^4}{r-r^5\omega_0^4}F_1^{(i)}&=\frac{(-1)^{i+1}(3-5r^4\omega_0^4+45r^8\omega_0^8+5r^{12}\omega_0^{12})}{2r^2\omega_0(1-r^4\omega_0^4)^{7/2}}. \quad (i=3,4)\end{aligned} \tag{92}$$

The particular integrals of Eq. (92) are

$$F_1^{(i)}(r)=\begin{cases}(-1)^i3r^2\omega_0(1+r^4\omega_0^4)/(2(1-r^4\omega_0^4)^{5/2}), i=1,2\\(-1)^i(3-10r^4\omega_0^4-5r^8\omega_0^8)/(4\,r\,\omega_0(1-r^4\omega_0^4)^{5/2}).\ \ i=3,4\end{cases} \tag{93}$$

By expanding Eq. (61) and in view of relations Eq. (89), Eq. (91) and Eq. (93), we obtain

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$$\lambda_a = 0.543689 + \frac{0.385922}{A}\frac{1}{\mu} - \frac{4.184334}{A^2}\frac{1}{\mu^2} + \cdots,$$
$$\omega_0 = \frac{0.771845}{A} - \frac{1.305566}{A^2}\frac{1}{\mu} + \frac{15.396644}{A^3}\frac{1}{\mu^2} + \cdots. \quad (94)$$

For the special case $\frac{A}{L} = 1$, the above results are the same as what Coman and Destrade [24] obtained. Our WKB curves plotted in Fig. 2 are exactly similar to the curves of Fig. 5 (of the later article) which are obtained with the help of compound matrix method.

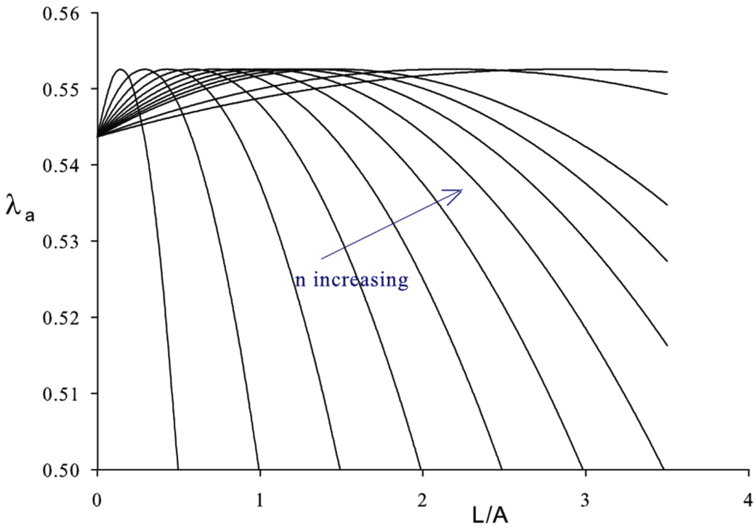

Fig. 2. A WKB plot of the critical values of λ_a against L/A for a Neo-Hookean cube and for mode numbers n = 1, 2, ..., 10, 15, 20.

The comparisons between the compound matrix and WKB results for λ_a and φ (where $\frac{A}{L} = \frac{1}{2}, \frac{1}{3}$) are given in Table 3. It is worth noting that, the existence of the turning point is the main reason of the tiny difference between the results of the two methods. For example we checked an special case i.e. $n = 20$ and $\frac{A}{L} = 1$, which yield the turning point $r = 1.32456$.

Table 3. Comparisons between numerical and asymptotic results of the azimuthal stretch and bending angle related to Neo-Hookean cube, for $\frac{A}{L} = \frac{1}{2}, \frac{1}{3}$ and different mode numbers.

	$\frac{A}{L} = \frac{1}{2}$				$\frac{A}{L} = \frac{1}{3}$			
CMM		**WKB**			**CMM**		**WKB**	
n	λ_a	φ^0	λ_a	φ^0	λ_a	φ^0	λ_a	φ^0
3.0	0.557579	166.48	0.437157	0.437157	0.542259	267.009	0.24257	0.24257
4.0	0.560907	164.087	0.49912	0.49912	0.554789	252.775	0.397343	0.397343
5.0	0.56135	163.771	0.524992	0.524992	0.559279	247.879	0.464769	0.464769
6.0	0.560907	164.087	0.53753	0.53753	0.560907	246.13	0.49912	0.49912
7.0	0.560174	164.611	0.544178	0.544178	0.561353	245.654	0.518465	0.518465
8.0	0.559367	165.19	0.547902	0.547902	0.561258	245.754	0.530135	0.530135

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9.0	0.558568	165.765	0.550051	0.550051	0.560907	246.131	0.53753	0.53753
10.0	0.557809	166.313	0.551299	0.551299	0.560433	246.638	0.542385	0.542385
12.0	0.556452	167.299	0.552386	0.552386	0.559367	247.785	0.547902	0.547902
14.0	0.555301	168.141	0.552586	0.552586	0.55831	248.927	0.550545	0.550545
16.0	0.554326	168.857	0.55242	0.55242	0.557333	249.988	0.551817	0.551817
18.0	0.553495	169.47	0.552104	0.552104	0.556452	250.949	0.552386	0.552386
20.0	0.55278	170.0	0.551734	0.551734	0.555664	251.813	0.552576	0.552576

WKB analysis of pure bending of a Varga cube

This problem recently has been solved as a system of equations by Sanjaranipour, Hatami and Abdolalian [26]. As we mentioned in the previous sections and regarding to our goal, i.e. analyzing and comparing the differential equations of $F_0(r)$, $F_1(r)$, $F_2(r)$, ..., we are solving here the fourth order differential equation instead of a system of equations. In light of the procedure of the previous section we write dwon here a brief description. To the leading order two repeated roots of $S(r)$ are

$$S^{(1)}(r) = S^{(3)}(r) = 1, S^{(2)}(r) = S^{(4)}(r) = -1. \tag{95}$$

By using Eq. (95), the next order term (i.e. $O(\mu^3)$) is automatically satisfied. By continuing the analysis to the next order (i.e. $O(\mu^2)$), we obtain the following second order differential equation of $F_0(r)$

$$(F_0^{(i)})'' - \frac{2\,r\,\omega_0^2}{1 + r^2\omega_0^2}(F_0^{(i)})' + \frac{3\omega_0^2 - r^2\omega_0^4}{4(1 + r^2\omega_0^2)}F_0^{(i)} = 0. (i = 1,2,3,4) \tag{96}$$

Two independent solutions of Eq. (96) are

$$F_0^{(i)}(r) = \begin{cases} 2(1 + r\,\omega_0)\exp\{-(1 + r\,\omega_0)/2\}, & i = 1,2 \\ 2(-1 + r\,\omega_0)\exp\{-(1 - r\,\omega_0)/2\}, & i = 3,4 \end{cases} \tag{97}$$

where superscripts here correspond to Eq. (95). By equating the coefficient of μ and with the use of Eq. (95) and Eq. (97), four second order inhomogeneous differential equations for $F_1(r)$, yield

$$(F_1^{(i)})'' - \frac{2\,r\,\omega_0^2}{1 + r^2\omega_0^2}(F_1^{(i)})' + \frac{3\omega_0^2 - r^2\omega_0^4}{4(1 + r^2\omega_0^2)}F_1^{(i)} = \begin{cases} \dfrac{(-1)^i\omega_0^3(5 + r\,\omega_0 - r^2\omega_0^2 - r^3\omega_0^3)}{4(1 + r^2\omega_0^2)}\exp\{\dfrac{-(1 + r\,\omega_0)}{2}\}, & i = 1,2 \\ \dfrac{(-1)^i\omega_0^3(5 - r\,\omega_0 - r^2\omega_0^2 + r^3\omega_0^3)}{4(1 + r^2\omega_0^2)}\exp\{\dfrac{-(1 - r\,\omega_0)}{2}\}. & i = 3,4 \end{cases} \tag{98}$$

The particular integrals of the differential equation of Eq. (98), which can be find easily by using the so-called reduction of order method, are

$$F_1^{(i)}(r) = \begin{cases} \dfrac{(-1)^i\omega_0}{4}(5 + 2\,r\,\omega_0 + r^2\omega_0^2)\exp\{-(1 + r\,\omega_0)/2\}, & i = 1,2 \\ \dfrac{(-1)^i\omega_0}{4}(5 - 2\,r\,\omega_0 + r^2\omega_0^2)\exp\{-(1 - r\,\omega_0)/2\}, & i = 3,4 \end{cases} \tag{99}$$

and by continuing the same process as before, we obtain

$$F_2^{(i)}(r) = \begin{cases} \dfrac{-\omega_0^2(22 + 12\,r\,\omega_0 + 5r^2\omega_0^2 - r^3\omega_0^3)}{64}\exp\{-(1 + r\,\omega_0)/2\}, & i = 1,2 \\ \dfrac{\omega_0^2(22 - 12\,r\,\omega_0 + 5r^2\omega_0^2 + r^3\omega_0^3)}{64}\exp\{-(1 - r\,\omega_0)/2\}. & i = 3,4 \end{cases} \tag{100}$$

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For the special case $\frac{A}{L}=\frac{1}{3}$, the above results are the same as what Sanjaranipour, Hatami and Abdolalian [26] obtained. In Fig. 3, we have shown the comparison between the numerical and asymptotic results for n = 2, 3, ..., 10, 15, 20.

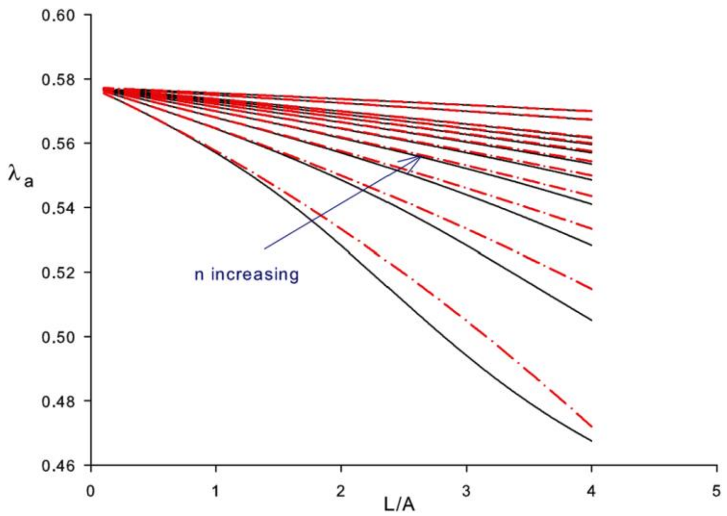

Fig. 3. Comparisons between the curves of the numerical and asymptotic results for the critical values of λ_a against L/A related to Varga cube for n = 2, 3, ..., 10, 15, 20. Solid lines: numerical results; dashed dotted lines represent the asymptotic results.

Comparing between the compound matrix and WKB results for λ_a and φ (where $\frac{A}{L}=1,\frac{1}{2},\frac{1}{3}$) are given in Table 4 and Table 5. Also the results of Table 5 are similar to the results of Table 1 and Table 2 (for $\frac{A}{L}=\frac{1}{2},\frac{1}{3}$) in the article of Sanjaranipour, Hatami and Abdolalian [26]. Indeed the asymptotic results approximate the numerical results extremely well over the region of their validity and this is true since WKB approximation give better and precise results for high mode numbers.

Table 4. The comparisons between the numerical and asymptotic results of the azimuthal stretch λ_a and the bending angle φ related to Varga cube, for $\frac{A}{L}=1$ and different mode numbers.

	CMM		WKB			CMM		WKB	
n	λ_a	φ^0	λ_a	φ^0	n	λ_a	φ^0	λ_a	φ^0
1.0	0.5282	94.69	0.533316	91.893	11.0	0.574061	77.491	0.574063	77.489
2.0	0.55698	83.458	0.5575	83.194	12.0	0.574341	77.397	0.574343	77.396
3.0	0.564458	80.787	0.564598	80.716	13.0	0.574577	77.318	0.574578	77.317
4.0	0.56791	79.585	0.567967	79.557	14.0	0.574779	77.25	0.57478	77.25
5.0	0.569902	78.9	0.56993	78.886	15.0	0.574953	77.192	0.574954	77.191
6.0	0.571199	78.458	0.571215	78.45	16.0	0.575105	77.141	0.575106	77.141
7.0	0.572111	78.148	0.572121	78.143	17.0	0.575239	77.096	0.57524	77.096
8.0	0.572787	77.92	0.572794	77.916	18.0	0.575358	77.056	0.575359	77.056
9.0	0.573309	77.744	0.573314	77.741	19.0	0.575464	77.021	0.575465	77.021
10.0	0.573723	77.604	0.573727	77.602	20.0	0.57556	76.989	0.57556	76.989

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Table 5. The comparisons between the numerical and asymptotic results of the azimuthal stretch λ_a and the bending angle φ related to Varga cube, for $\frac{A}{L}=\frac{1}{2},\frac{1}{3}$ and for different mode numbers.

	$\frac{A}{L}=\frac{1}{2}$				$\frac{A}{L}=\frac{1}{3}$			
	CMM		WKB		CMM		WKB	
n	λ_a	φ^0	λ_a	φ^0	λ_a	φ^0	λ_a	φ^0
2.0	0.5282	189.38	0.533316	183.785	0.494045	331.134	0.504799	307.471
4.0	0.55698	166.915	0.5575	166.388	0.543913	265.081	0.54595	261.917
6.0	0.564458	161.574	0.564598	161.433	0.55698	250.373	0.5575	249.581
8.0	0.56791	159.17	0.567967	159.113	0.562665	244.257	0.562869	243.948
10.0	0.569902	157.801	0.56993	157.773	0.56586	240.89	0.56596	240.738
12.0	0.571199	156.916	0.571215	156.9	0.56791	238.755	0.567967	238.67
14.0	0.572111	156.297	0.572121	156.287	0.569339	237.28	0.569374	237.228
16.0	0.572787	155.839	0.572794	155.832	0.570391	236.2	0.570414	236.165
18.0	0.573309	155.487	0.573314	155.482	0.571199	235.374	0.571215	235.35
20.0	0.573723	155.208	0.573727	155.205	0.571838	234.722	0.57185	234.704

Expanding Eq. (61) with the help of relations Eq. (95), Eq. (97), Eq. (99) and Eq. (100), yield

$$(1-3\lambda_a^2)-\frac{\lambda_a(1+\lambda_a^2)}{2\,A}\frac{1}{\mu}-\frac{(1+\lambda_a^2)^2}{128A^2\lambda_a^4}(3-11\lambda_a^2+37\lambda_a^4-13\lambda_a^6)\frac{1}{\mu^2}+O(\frac{1}{\mu^3})=0, \tag{101}$$

where $\lambda_a=r\,\omega_0$. By expanding λ_a and ω_0 in term of $\frac{1}{\mu}$, we ge

$$\begin{aligned}\lambda_a&=0.57735-\frac{0.11111}{A}\frac{1}{\mu}-\frac{0.085533}{A^2}\frac{1}{\mu^2}+\cdots,\\ \omega_0&=\frac{0.666667}{A}+\frac{0.32075}{A^2}\frac{1}{\mu}+\frac{0.327161}{A^3}\frac{1}{\mu^2}+\cdots.\end{aligned} \tag{102}$$

WKB analysis of the buckling of a Neo-Hookean spherical shell

This problem has been solved previously with the help of the standard WKB expansion by Fu [29]. Again regarding to our goal in this article, we continue the same analysis as before (i.e. applying the specific form of WKB expansion Eq. (47)) and obtain the following four un-repeated roots for $S(r)$

$$S^{(1)}(r)=\frac{1}{r},S^{(2)}(r)=\frac{-1}{r},S^{(3)}(r)=\frac{\lambda^3}{r},S^{(4)}(r)=\frac{-\lambda^3}{r}. \tag{103}$$

By substituting $S(r)$ and $\lambda=\frac{r}{(q+r^3)^{1/3}}$ in the next order terms (where λ is obtained with the use of Eq. (35) and Eq. (36) and also $q=A^3-a^3$), we obtain the following first order differential equations of $F_0(r)$

$$\begin{aligned}&(F_0^{(1)})'-\frac{2q^2+4\,q\,r^3+3r^6}{r\,(q+r^3)(q+2r^3)}F_0^{(1)}=0,\quad (F_0^{(2)})'-\frac{q^2+q\,r^3+r^6}{r\,(q+r^3)(q+2r^3)}F_0^{(2)}=0,\\ &(F_0^{(3)})'-\frac{3r^5}{(q+r^3)(q+2r^3)}F_0^{(3)}=0,\quad (F_0^{(4)})'+\frac{r^2(q-r^3)}{(q+r^3)(q+2r^3)}F_0^{(4)}=0,\end{aligned} \tag{104}$$

where superscripts correspond to Eq. (103). Then four solutions of $F_0(r)$ are

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$$F_0^{(1)}(r)=\frac{r^2(q+r^3)^{1/3}}{\sqrt{q+2r^3}},F_0^{(2)}(r)=\frac{r(q+r^3)^{1/3}}{\sqrt{q+2r^3}},$$
$$F_0^{(3)}(r)=\frac{q+r^3}{\sqrt{q+2r^3}},F_0^{(4)}(r)=\frac{(q+r^3)^{2/3}}{\sqrt{q+2r^3}}. \tag{105}$$

Accordingly, four inhomogeneous differential equations of $F_1(r)$ are given by

$$(F_1^{(1)})'-\frac{2q^2+4\,q\,r^3+3r^6}{r(q+r^3)(q+2r^3)}F_1^{(1)}=\frac{r^4(24q^5+69q^4r^3+62q^3r^6+3q^2r^9-30\,q\,r^{12}-20r^{15})}{2\,q(q+2r^3)^{7/2}(q+r^3)^{5/3}},$$
$$(F_1^{(2)})'-\frac{q^2+q\,r^3+r^6}{r(q+r^3)(q+2r^3)}F_1^{(2)}=\frac{-r^3(24q^5+69q^4r^3+62q^3r^6+3q^2r^9-30\,q\,r^{12}-20r^{15})}{2\,q(q+2r^3)^{7/2}(q+r^3)^{5/3}},$$
$$(F_1^{(3)})'-\frac{3r^5}{(q+r^3)(q+2r^3)}F_1^{(3)}=\frac{10q^6+64q^5r^3+149q^4r^6+114q^3r^9+11q^2r^{12}+10\,q\,r^{15}+20r^{18}}{2\,q\,r^4(q+2r^3)^{7/2}}, \tag{106}$$
$$(F_1^{(4)})'+\frac{r^2(q-r^3)}{q^2+3\,q\,r^3+2r^6}F_1^{(4)}=-\frac{10q^6+64q^5r^3+149q^4r^6+114q^3r^9+11q^2r^{12}+10\,q\,r^{15}+20r^{18}}{2\,q\,r^4(q+2r^3)^{7/2}(q+r^3)^{1/3}},$$

and the particular integrals are, respectively

$$F_1^{(i)}(r)=\frac{(-1)^i r^{3-i}(q+r^3)^{1/3}\{\frac{31q^4+27q^3r^3+44q^2r^6+160\,q\,r^9+80r^{12}}{8\,q(q+r^3)(q+2r^3)^2}-\ln[(q+r^3)^2(q+2r^3)^3]\}}{6\sqrt{q+2r^3}},$$
$$(i=1,2) \tag{107}$$
$$F_1^{(i)}(r)=\frac{(-1)^i(q+r^3)^{\frac{6-i}{3}}\{\frac{80q^4+311q^3r^3+228q^2r^6-80\,q\,r^9-80r^{12}}{8\,q\,r^3(q+2r^3)^2}+\ln[\frac{r^{18}(q+r^3)^2}{(q+2r^3)^3}]\}}{6\sqrt{q+2r^3}}.(i=3,4)$$

Now, by expanding Eq. (61) with the use of Eq. (103), Eq. (105) and Eq. (107), yields

$$\lambda_a=\frac{a}{A}=0.666142+0.590915/n-10.267945/n^2+\cdots. \tag{108}$$

This expansion is exactly the same as what obtained by Fu [30].

Asymptotic results for $A-1=O(\frac{1}{n})$

Similar to previous section we consider expansions Eq. (64) and Eq. (65) for A and λ_a and substitute into relations $a=\lambda_a\,A$ and $b=\lambda_b$ and obtain

$$a=\eta_1+(\xi\eta_1+\eta_2)\frac{1}{n}+\cdots,\quad b=\eta_1+\frac{-\xi+\xi\eta_1^2+\eta_1\eta_2}{\eta_1^2}\frac{1}{n}+\cdots. \tag{109}$$

By substituting a and b from Eq. (109) into $M=(M_{ij})$ and then by expanding $\det(M_{ij})$, with the aid of Mathematica, we find that, to the leading order, $\det(M_{ij})=0$, gives

$$4(1+20z^2+6z^4+4z^6+z^8)\sinh(\xi)\sinh(\frac{\xi}{z})$$
$$-32z(z^2+1)^2(-1+\cosh(\xi)\cosh(\frac{\xi}{z}))=0, \tag{110}$$

where $z=\eta_1^3$. The procedure of obtaining η_2 and λ_a is the same as what has been done in previous section. In Fig. 4, λ_a of the previous and this section (λ_a of the outer and inner layer, respectively) are plotted together, which are compared with the results of the compound matrix method and are exactly the same as the results obtained by Fu [29] in Fig. 2 and Fig. 3.

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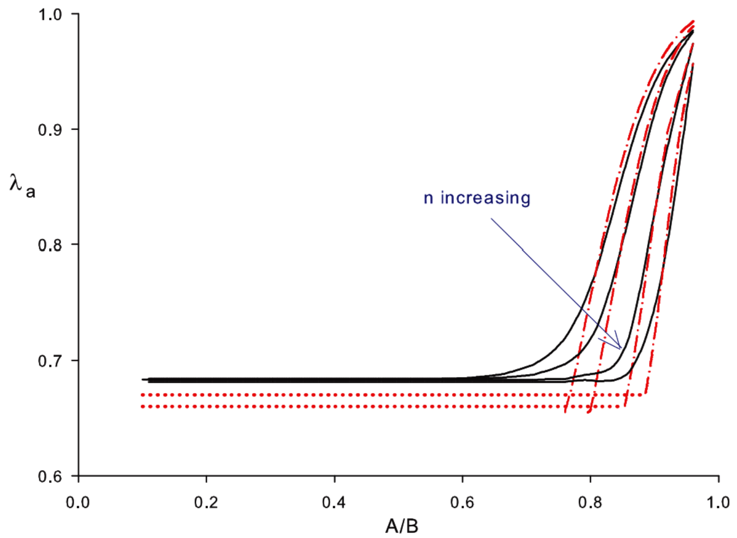

Fig. 4. Bifurcation curves of Neo-Hookean spherical shell subjected to an external hydrostatic pressure for n =8, 10, 15, 20. Solid lines: numerical results; dotted lines: WKB outer layer results; dashed dotted lines represent the WKB inner layer results.

WKB analysis of the eversion of a Neo-Hookean spherical shell

In this section again we apply the previous WKB expansion i.e. Eq. (47) and continue the same analysis as before. Hence, by doing so, four un-repeated roots of $S(r)$, yields

$$S^{(1)}(r)=\frac{1}{r},S^{(2)}(r)=\frac{-1}{r},S^{(3)}(r)=\frac{\lambda^3}{r},S^{(4)}(r)=\frac{-\lambda^3}{r}. \tag{111}$$

In view of $S(r)$ and $\lambda=\frac{r}{(q-r^3)^{1/3}}$, we obtain the following four first order ordinary differential equations of $F_0(r)$

$$\begin{aligned}&(F_0^{(1)})'-\frac{2q^2-4\,q\,r^3+3r^6}{r\,(q-r^3)(q-2r^3)}F_0^{(1)}=0,(F_0^{(2)})'-\frac{q^2-q\,r^3+r^6}{r\,(q-r^3)(q-2r^3)}F_0^{(2)}=0,\\&(F_0^{(3)})'-\frac{r^2(q+r^3)}{r\,(q-r^3)(q-2r^3)}F_0^{(3)}=0,(F_0^{(4)})'-\frac{3r^5}{r\,(q-r^3)(q-2r^3)}F_0^{(4)}=0,\end{aligned} \tag{112}$$

where $q=1+a^3$. The relevant solutions of Eq. (112), are respectively

$$\begin{aligned}&F_0^{(1)}(r)=\frac{r^2(q-r^3)^{1/3}}{\sqrt{q-2r^3}},F_0^{(2)}(r)=\frac{r\,(q-r^3)^{1/3}}{\sqrt{q-2r^3}},\\&F_0^{(3)}(r)=\frac{(q-r^3)^{2/3}}{\sqrt{q-2r^3}},F_0^{(4)}(r)=\frac{q-r^3}{\sqrt{q-2r^3}},\end{aligned} \tag{113}$$

where superscripts here correspond to those in Eq. (111). The same procedure as before has been employed and we obtained the following four first order inhomogeneous differential equations of $F_1(r)$,

$$\begin{aligned}&(F_1^{(1)})'-\frac{2q^2-4\,q\,r^3+3r^6}{r\,(q-r^3)(q-2r^3)}F_1^{(1)}=\frac{-r^4(24q^5-69q^4r^3+62q^3r^6-3q^2r^9-30\,q\,r^{12}+20r^{15})}{2\,q\,(q-2r^3)^{7/2}(q-r^3)^{5/3}},\\&(F_1^{(2)})'-\frac{q^2-q\,r^3+r^6}{r\,(q-r^3)(q-2r^3)}F_1^{(2)}=\frac{r^3(24q^5-96q^4r^3+62q^3r^6-3q^2r^9-30\,q\,r^{12}+20r^{15})}{2\,q\,(q-2r^3)^{7/2}(q-r^3)^{5/3}},\\&(F_1^{(3)})'-\frac{r^2(q+r^3)}{r\,(q-r^3)(q-2r^3)}F_1^{(3)}=\frac{10q^6-64q^5r^3+149q^4r^6-114q^3r^9+11q^2r^{12}-10\,q\,r^{15}+20r^{18}}{2\,q\,r^4(q-2r^3)^{7/2}(q-r^3)^{1/3}},\\&(F_1^{(4)})'-\frac{3r^5}{r\,(q-r^3)(q-2r^3)}F_1^{(4)}=\frac{-10q^6+64q^5r^3-149q^4r^6+114q^3r^9-11q^2r^{12}+10\,q\,r^{15}-20r^{18}}{2\,q\,r^4(q-2r^3)^{7/2}},\end{aligned} \tag{114}$$

and the particular integrals of Eq. (114) are respectively

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$$F_1^{(i)}(r) = \frac{(-r)^{3-i}(q-r^3)^{\frac{1}{3}}\{\frac{-31q^4+27q^3r^3-44q^2r^6+160\,q\,r^9-80r^{12}}{8\,q\,(q-r^3)(q-2r^3)^2} + \ln((q-r^3)^2(q-2r^3)^3)\}}{6\sqrt{q-2r^3}}, \quad (i=1,2)$$

$$F_1^{(i)}(r) = \frac{(-1)^{i+1}(q-r^3)^{(5-i)/3}\{\frac{-80q^4+311q^3r^3-228q^2r^6-80\,q\,r^9+80r^{12}}{8\,q\,r^3(q-2r^3)^2} + \ln(\frac{r^{18}(q-r^3)^2}{(q-2r^3)^3})\}}{6\sqrt{q-2r^3}}. \quad (i=3,4) \tag{115}$$

By expanding Eq. (61) of course with the use of Eq. (111), Eq. (113) and Eq. (115), the final result of λ_a can be written as

$$\lambda_a = a = 0.666142 + 1.377044/n - 23.289856/n^2 + \cdots. \tag{116}$$

In view of the relations Eq. (40) and Eq. (42), we have

$$A = 0.252188 + 8.148439/n - 187.262476/n^2 + \cdots. \tag{117}$$

Table 6 shows the comparisons between the compound matrix and WKB results for critical values of $\frac{A}{B}$ and the corresponding values of $\frac{a}{B}$, $\frac{b}{B}$ for various mode numbers. It seems that the tiny difference between the results of the two mentioned methods is the existence of the turning point. For example, we checked a special case i.e. $n = 20$, which yields the turning point at $r = 0.868449$.

Table 6. The critical values of $\frac{A}{B}$ and the corresponding values of $\frac{a}{B}$ and $\frac{b}{B}$ for various mode numbers related to Neo-Hookean everted sphere.

	CMM			WKB		
n	$\frac{A}{B}$	$\frac{a}{B}$	$\frac{b}{B}$	$\frac{A}{B}$	$\frac{a}{B}$	$\frac{b}{B}$
16.0	0.449435	0.705574	1.080219	0.029972	0.661232	1.08833
17.0	0.445607	0.704665	1.080488	0.083541	0.666557	1.09015
18.0	0.441909	0.703793	1.080742	0.126909	0.670763	1.09132
19.0	0.438305	0.70295	1.080982	0.162321	0.674104	1.09197
20.0	0.43478	0.702131	1.081211	0.191454	0.67677	1.09222
25.0	0.418292	0.698375	1.082201	0.278506	0.68396	1.09093
50.0	0.363472	0.686702	1.084579	0.340252	0.684367	1.08609
100.0	0.318432	0.677928	1.085563	0.314947	0.677584	1.08573
150.0	0.299296	0.674396	1.085742	0.298189	0.674288	1.08578
200.0	0.288734	0.672493	1.085784	0.288249	0.672445	1.0858
250.0	0.28204	0.671303	1.085791	0.281786	0.671278	1.0858
300.0	0.277419	0.670488	1.085787	0.277269	0.670474	1.08579
350.0	0.274036	0.669896	1.085779	0.273941	0.669887	1.08578
400.0	0.271453	0.669446	1.085771	0.271389	0.669439	1.08577
450.0	0.269417	0.669092	1.085763	0.269371	0.669087	1.08576

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500.0	0.267769	0.668807	1.085756	0.267736	0.668803	1.08576
550.0	0.26641	0.668572	1.085749	0.266385	0.668569	1.08575
600.0	0.265268	0.668375	1.085742	0.265249	0.668373	1.08574
650.0	0.264297	0.668207	1.085737	0.264281	0.668206	1.08574
700.0	0.263459	0.668063	1.085732	0.263447	0.668062	1.08573

WKB analysis for the eversion of a Varga spherical shell

This problem has been solved previously by Haughton and Chen [36]. They applied the special form of WKB expansion i.e. Eq. (47) and obtained just leading order terms. Here, we continue the analysis to the higher orders. The same procedure as before has been employed and the following four repeated roots of $S(r)$ are obtained

$$S^{(1)}(r) = S^{(3)}(r) = \frac{\lambda^{3/2}}{r}, S^{(2)}(r) = S^{(4)}(r) = \frac{-\lambda^{3/2}}{r}. \tag{118}$$

By applying $S(r)$ from Eq. (118), the next order terms are automatically satisfied. Upon using Eq. (118) in the new order, we obtain the following two second order differential equations of $F_0(r)$

$$\begin{aligned} &(F_0^{(i)})'' + \frac{(3-\lambda^{3/2}-2\lambda^3)}{r}(F_0^{(i)})' + \frac{(7-28\lambda^{3/2}-22\lambda^3+4\lambda^{9/2}-5\lambda^6)}{16r^2}F_0^{(i)} = 0, \quad (i=1,3) \\ &(F_0^{(i)})'' + \frac{(3+\lambda^{3/2}-2\lambda^3)}{r}(F_0^{(i)})' + \frac{(7+28\lambda^{3/2}-22\lambda^3-4\lambda^{9/2}-5\lambda^6)}{16r^2}F_0^{(i)} = 0. \quad (i=2,4) \end{aligned} \tag{119}$$

In order to be able to solve the equations Eq. (119), we use $\frac{d\lambda}{dr} = \frac{\lambda(1+\lambda^3)}{r}$ (given by Haughton and chen [36]) and obtain the following simplified differential equations of $F_0(\lambda)$

$$\begin{aligned} &(F_0^{(i)})'' + \frac{3-\lambda^{3/2}+2\lambda^3}{\lambda(1+\lambda^3)}(F_0^{(i)})' + \frac{7-28\lambda^{3/2}-22\lambda^3+4\lambda^{9/2}-5\lambda^6}{16\lambda^2(1+\lambda^3)^2}F_0^{(i)} = 0, \quad (i=1,3) \\ &(F_0^{(i)})'' + \frac{3+\lambda^{3/2}+2\lambda^3}{\lambda(1+\lambda^3)}(F_0^{(i)})' + \frac{7+28\lambda^{3/2}-22\lambda^3-4\lambda^{9/2}-5\lambda^6}{16\lambda^2(1+\lambda^3)^2}F_0^{(i)} = 0, \quad (i=2,4) \end{aligned} \tag{120}$$

and the solutions are,

$$F_0^{(i)}(\lambda) = \begin{cases} \lambda^{-7/4}(1+\lambda^3)^{1/6}\exp((-1)^i\, d/3), & i=1,2 \\ \frac{2}{3}\lambda^{-1/4}(1+\lambda^3)^{1/6}\exp((-1)^i\, d/3), & i=3,4 \end{cases} \tag{121}$$

where $d = \arctan(\sqrt{3}-2\sqrt{\lambda}) - \arctan(\sqrt{3}+2\sqrt{\lambda}) + \arctan(\sqrt{\lambda})$. By continuing the process of the previous sections and in view of Eq. (118) and Eq. (121) four second order inhomogeneous differential equations of $F_1(\lambda)$ are obtained

$$\begin{aligned} &(F_1^{(i)})'' + \frac{3+(-1)^i\lambda^{3/2}+2\lambda^3}{\lambda(1+\lambda^3)}(F_1^{(i)})' + \frac{7+(-1)^i 28\lambda^{3/2}-22\lambda^3+(-1)^{i+1}4\lambda^{9/2}-5\lambda^6}{16\lambda^2(1+\lambda^3)^2}F_1^{(i)} \\ &= \begin{cases} \frac{(-1)^i}{32}\lambda^{-21/4}(1+\lambda^3)^{-11/6}(21+42\lambda^3-99\lambda^6)\exp(\frac{(-1)^i\, d}{3}), & i=1,2 \\ \frac{(-1)^i}{16}\lambda^{-3/4}(1+\lambda^3)^{-11/6}(53+26\lambda^3+13\lambda^6)\exp(\frac{(-1)^i\, d}{3}), & i=3,4 \end{cases} \end{aligned} \tag{122}$$

and the particular integrals which can be found easily by using the reduction of order method, are

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$$F_1^{(i)}(\lambda) = \begin{cases} \dfrac{(-1)^i(1+\lambda^3)^{\frac{1}{6}}\exp((-1)^i\, d/3)}{48\lambda^{\frac{13}{4}}}(7-40\lambda^{\frac{3}{2}}\, arccot(\lambda^{\frac{3}{2}})), & i=1,2 \\ \dfrac{(-1)^i(1+\lambda^3)^{\frac{1}{6}}\exp((-1)^i\, d/3)}{72\lambda^{7/4}}(27+13\lambda^3-40\lambda^{\frac{3}{2}}\, arccot(\lambda^{\frac{3}{2}})). & i=3,4 \end{cases} \quad (123)$$

Now we expand Eq. (61) with the use of Eq. (118), Eq. (121) and Eq. (123), and get

$$\lambda_a = 0.693361 - 0.088958/n - 0.198055/n^2 + \cdots. \quad (124)$$

In view of relation Eq. (41), we have

$$A = 0.46792 - 0.372274/n - 1.055095/n^2 + \cdots. \quad (125)$$

The following numerical results which obtained with the aid of compound matrix method are also similar to those of Haughton and Chen [35]. The comparison between the critical values of $\frac{A}{B}$ and the corresponding values of $\frac{a}{B}$ and $\frac{b}{B}$ for various mode numbers are given in Table 7. It is obvious from the table that the results of the two methods are almost coincident and also as the mode number increases the results getting closer and closer.

Table 7. The critical values of $\frac{A}{B}$ and the corresponding values of $\frac{a}{B}$ and $\frac{b}{B}$ for various mode numbers related to Varga everted sphere.

	CMM			WKB		
n	$\frac{A}{B}$	$\frac{a}{B}$	$\frac{b}{B}$	$\frac{A}{B}$	$\frac{a}{B}$	$\frac{b}{B}$
4.0	0.308908	0.658743	1.07905	0.420449	0.682861	1.075518
6.0	0.376566	0.673033	1.07764	0.416114	0.681983	1.07582
8.0	0.4049	0.679147	1.07632	0.420209	0.682813	1.075535
10.0	0.420142	0.682485	1.07541	0.427076	0.684229	1.075041
12.0	0.42957	0.684573	1.07478	0.433339	0.68555	1.074574
14.0	0.435946	0.685997	1.07433	0.438266	0.686609	1.074195
16.0	0.440532	0.687028	1.07399	0.442079	0.687441	1.073895
18.0	0.443982	0.687808	1.07372	0.44507	0.688101	1.073655
20.0	0.446669	0.688418	1.07351	0.447465	0.688634	1.073461
25.0	0.451341	0.689486	1.07313	0.451754	0.689599	1.073106
100.0	0.464092	0.692452	1.07204	0.464099	0.692454	1.072041
150.0	0.465392	0.692759	1.07193	0.465394	0.69276	1.071925
200.0	0.466033	0.692912	1.07187	0.466034	0.692912	1.071868
300.0	0.466668	0.693063	1.07181	0.466668	0.693063	1.071811
400.0	0.466983	0.693138	1.07178	0.466983	0.693138	1.071782
500.0	0.467172	0.693183	1.07177	0.467172	0.693183	1.071765
600.0	0.467297	0.693212	1.07175	0.467297	0.693212	1.071754
700.0	0.467386	0.693234	1.07175	0.467386	0.693234	1.07175

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CONCLUSION

Apart from solving the eigenvalue problems (fourth order simplified ODE,s) of the cylinder, everted cylinder, sphere, everted sphere and cube, by using the specific form of WKB expansion i.e. $F(r) = (F_0(r) + \frac{1}{n}F_1(r) + \frac{1}{n^2}F_2(r) + \frac{1}{n^3}F_3(r) + \cdots)exp\left(n\int_a^r S(r)dr\right)$, our aim is to find, solve and compare the differential equations obtained for $F_0(r)$, $F_1(r)$, $F_2(r)$, ... and also the roots of $S(r)$. In order to do so, we extended our analysis to one order higher than what has been previously done (the leading or/and first order). We noticed that, the differential equations obtained for $F_0(r)$, $F_1(r)$, $F_2(r)$, ..., (i.e. Eq. (70), Eq. (72), ..., Eq. (120), Eq. (122)) are second order for Varga, while are first order (i.e. Eq. (50), Eq. (52), ..., Eq. (112), Eq. (114)) for Neo-Hookean materials. By applying this specific form of WKB expansion to the relevant eigenvalue problems and by collecting the leading order terms, it is confirmed that the roots of $S(r)$ are repeated for Varga (see Eq. (69), Eq. (81), Eq. (95), Eq. (118)) and un-repeated for Neo-Hookean materials (e.g. Eq. (49), Eq. (74), Eq. (89), Eq. (103), Eq. (111)). Moreover, based on the analysis and due to the results obtained, we found that $S(r)$ and the ODE,s of $F_0(r)$, $F_1(r)$, $F_2(r)$, ... depends on the elastic material but not to the relevant geometry.

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