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**Research Article**

# 3-Equitable Ceiling Average Labeling

Slashi Leel [1], Sweta Srivastav [2], Sangeeta Gupta [3]
[1,2,3] *Department of Mathematics, Sharda University, Greater Noida, Uttar Pradesh 201310, India*
[1] *2022301198.slashi@dr.sharda.ac.in*
[2] *sweta.srivastav@sharda.ac.in (Corresponding Author)*
[3] *sangeeta.gupta@sharda.ac.in*

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**ABSTRACT**

Let T = (V, E) represent a graph where V is the set of vertices and E is the set of edges. Consider a labeling of the vertices $f: V(T)\rightarrow\{0,1,2, \ldots, n-1\}$ such that each edge e=$uv$ is assigned a label $\lceil \frac{f(u)+f(v)}{2} \rceil \bmod 3$. Then we say graph T admits 3-equitable Ceiling Average Labeling if $|v_f(i)-v_f(j)|\leq 1$ and $|e_f(i)-e_f(j)|\leq 1$ for all $0 \leq i, j \leq 2$. A graph which admits 3-equitable Ceiling Average Labeling is called a 3-equitable Graph. In this paper we have discussed the labeling of cycle graph, path graph, star graph and wheel graph. This labeling pattern can be applied in Traffic Flow Optimization, Power Grid Balancing, Data Clustering in Machine Learning.

**Keywords:** 3-Equitable Labeling, Ceiling Function, Cycle Graph, Path Graph, Star Graph, Wheel Graph

## INTRODUCTION

The graphs discussed here are simple, finite, connected, and undirected. For all other terminology and notation, we follow Harrary [4]. Let T(V,E) be a graph where V(T) and E(T) represent the vertex set and edge set, respectively. When values are assigned to the vertices, edges, or both under specific conditions, this process is referred to as graph labeling. A comprehensive survey by Gallian [2] collects and updates many results concerning graph labelings, including cordial labelings.

Graph labeling has been a widely studied area in graph theory, with various forms and classifications being introduced over the years. One such labeling, known as 3-difference cordial labeling, was explored by Ponraj *et al.*[8], focusing on cycle-related graphs. Ghosh *et al.*[3] extended the study of graph labelings to 3-total sum cordial labeling, analyzing this labeling method on newly considered graphs like wheel, globe and a graph obtained by switching and duplication of arbitrary vertices of a cycle are 3-total sum cordial graphs. Srivastav *et al.* and Murugesan *et al.*[12, 5] further investigated the concept of 3-equitable prime cordial labeling, exploring its application on different types of graphs like Cycle with one chord, Cycle with twin chord, and Split Graph. Other significant contributions to this field include the work of Ponraj *et al.* [9], who introduced total prime cordial labeling of cycle-related graphs and Ali *et al.* [1], who examined total edge product cordial labeling in standard graphs namely wheel, gear and helm graphs. Comprehensive surveys on 3-equitable and divisor 3-equitable labeling, as well as prime cordial and divisor cordial labeling, were provided by Parthiban *et al.* [6, 7]. Furthermore, Srivastava *et al.* [11] presented new results on 3-equitable prime cordial labeling in certain graph classes using cyclic cubic graph, $P_n \cup C_n$ and wheel graphs, while Seenivasan *et al.* [10] studied vertex equitable labeling, contributing to the understanding of equitable labelings in graphs. A brief overview of these definitions is provided here, which will be useful for the current study.

**DEFINITION 1.1:** When values are assigned to vertices under specific conditions, it is referred to as graph labeling. Any form of graph labeling typically involves three key elements:

1) A defined set of numbers used for labeling the vertices;

2) A method for assigning a value to each edge;

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3) A requirement or condition that the edge values must meet.

**DEFINITION 1.2:** A ternary vertex labeling of a graph T is termed a 3-equitable labeling if $|v_f(i)-v_f(j)|\leq 1$ and $|e_f(i)-e_f(j)|\leq 1$ for all $0 \leq i, j \leq 2$. A graph that allows for 3-equitable Ceiling Average Labeling is known as a 3-equitable graph.

**DEFINITION 1.3:** Let T = (V, E) be a graph. A function $\delta : V(T)\rightarrow\{0, 1, 2\}$ is called a ternary vertex labeling of T and $\delta(v)$ represents the label of vertex v under $\delta$. For an edge e = uv, the corresponding induced edge labeling is defined by $\delta^* : E(T)\rightarrow\{0, 1, 2\}$. Let $v_\delta(0)$, $v_\delta(1)$, $v_\delta(2)$ denote the number of vertices in T labeled 0, 1, and 2, respectively under $\delta$ and $e_\delta(0)$, $e_\delta(1)$, $e_\delta(2)$ represent the number of edges labeled 0, 1, and 2, respectively under $\delta$.

**DEFINITION 1.4:** Let T be a (V,E) graph. A function $\delta : V(T)\rightarrow\{0,1,2, \ldots, n-1\}$is assigned to V. For each edge uv, we assign the label as $\lceil \frac{f(u)+f(v)}{2} \rceil \bmod 3$. The map $\delta$ is called 3-equitable Ceiling Average Labeling if $|v_f(i)-v_f(j)|\leq 1$ and $|e_f(i)-e_f(j)|\leq 1$ for all $0 \leq i, j \leq 2$.

**DEFINITION 1.5:** A cycle graph is a graph where vertices form a closed loop, with each vertex connected to two others. It's denoted as $C_n$, where n is the number of vertices.

**DEFINITION 1.6:** A path graph is a graph where vertices are arranged in a linear sequence, with each vertex connected to at most two others. It is represented as $P_n$, where n is the number of vertices.

**DEFINITION 1.7:** A star graph is a type of graph with one central vertex connected directly to all other vertices, which have no connections between them. It is represented as $S_n$, where n is the total number of vertices.

**DEFINITION 1.8:** A wheel graph is formed by connecting a single central vertex to all vertices of a cycle graph. It consists of a cycle with an additional hub vertex linked to every vertex in the cycle, creating a "wheel" shape. It is denoted as $W_n$, where n is the total number of vertices.

## MAIN RESULTS

**THEOREM 2.1:** The cycle graph $C_n$ preserves 3-equitable Ceiling Average Labeling $\forall$ $n \geq 4$.

**Proof:** Consider $a_1, a_2, a_3, \ldots\ldots, a_n$ the distinct vertices of a cycle graph $C_n$. The graph can be labeled according to a specific pattern involving the ceiling function and the mean of two consecutive vertex labels if:

- n mod 6 $\in\{0,1,4,5\}$ or
- n mod 6 $\in\{2,3\}$ then 0 is fixed at the first edge of the cycle graph.

CASE 1: when n mod 6 $\in\{0,1,4,5\}$

Vertex labeling will be done under the function $f$: $V(T) \longrightarrow \{0,1,2,\ldots, n-1\}$ as

$f(a_p) = p-1;\ 1 \leq p \leq n-1$

Let u and v are consecutive vertices of the cycle graph and an edge denoted by uv is induced between the vertex u and vertex v. Then the edge labeling will be done as follows under the function $f^*$

$f^*: E(T) \longrightarrow \{0,1,2\}$

$f^*(e=uv) = 0$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 0 (\bmod 3)$

$= 1$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 1 (\bmod 3)$

$= 2$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 2 (\bmod 3)$

CASE 2: when n mod 6 $\in\{2,3\}$

Vertex labeling will be done under the function $f$: $V(T) \longrightarrow \{0,1,2,\ldots, n-1\}$ as

$f(a_p) = p-1;\ 1 \leq p \leq n-1$

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Let $e_1$ be the first edge out of the n-edges in the $C_n$ graph. So when n mod 6 ∈{2,3} then the first edge $e_1$ will always be fixed to 0 i.e. $e_1 = 0$ and the remaining (n-1) edges will be mapped as follows:

$f^*$:E(T) - {$e_1$} –⟶ {0,1,2}

$f^*(e=uv) = 0$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 0 \pmod 3$

$= 1$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 1 \pmod 3$

$= 2$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 2 \pmod 3$

Hence, the cycle graph preserves 3-equitable Ceiling Average Labeling ∀ n ≥ 4

**Example:**

1) Cycle graph $C_7$ is a 3-equitable ceiling average labeling graph.

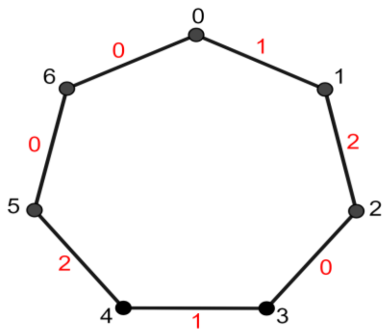


Figure 1: 3-equitable ceiling average labeling of $C_7$ graph

In Figure1, the red color numbers indicate the edge labeling.

2) Cycle graph $C_9$ is a 3-equitable ceiling average labeling graph.

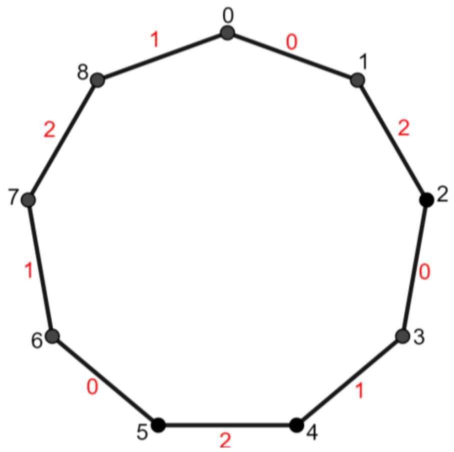


Figure 2: 3-equitable ceiling average labeling of $C_9$ graph

In Figure2, the red color numbers indicate the edge labeling.

**THEOREM 2.2:** The path graph $P_n$ preserves 3-equitable Ceiling Average Labeling ∀ n ≥ 4.

**Proof:** Let $a_1, a_2, a_3, \ldots\ldots\ldots\ldots, a_n$ represent the distinct vertices of a path graph $P_n$. The graph can be labeled following a particular pattern that uses the ceiling function along with the average of the labels of two consecutive vertices ∀ n ≥ 4.

Vertex labeling will be done under the function *f*: V(T) –⟶ {0,1,2,...., n-1} as

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$f(a_p) = p - 1; 1 \le p \le n-1$

Let u and v are consecutive vertices of the cycle graph and an edge denoted by uv is induced between the vertex u and vertex v. Then the edge labeling will be done as follows under the function $f^*$

$f^*$:E(T) –→ {0,1,2}

$f^*(e=uv) = 0$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 0 (mod3)$

$= 1$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 1 (mod3)$

$= 2$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 2 (mod3)$

Hence, the path graph preserves 3-equitable Ceiling Average Labeling ∀ n ≥ 4

**Example:**

1) Path graph $P_6$ is a 3-equitable ceiling average labeling graph.

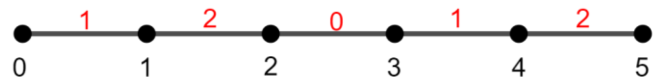


Figure 3: 3-equitable ceiling average labeling of $P_6$ graph

In Figure3, the red color numbers indicate the edge labeling.

2) Path graph $P_{11}$ is a 3-equitable ceiling average labeling graph.

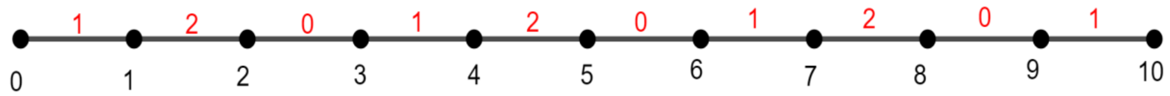


Figure 4: 3-equitable ceiling average labeling of $P_{11}$ graph

In Figure4, the red color numbers indicate the edge labeling.

**THEOREM 2.3:** The star graph $S_n$ preserves 3-equitable Ceiling Average Labeling ∀ n ≥ 6.

**Proof:** Let $a_1, a_2, a_3, \ldots\ldots, a_n$ be the distinct vertices of a star graph $S_n$. The graph is labeled based on a specific rule that involves the ceiling function and the average of the labels of two adjacent vertices if:

- n mod 6 ∈{0,1,2} or
- n mod 6 ∈{3,4,5} then 0 is fixed at the first edge of the cycle graph.

CASE 1: when n mod 6 ∈{0,1,2}

Vertex labeling will be done under the function $f$: V(T) –→ {0,1,2,...., n-1} as

$f(a_p) = p - 1; 1 \le p \le n-1$

Let u and v are consecutive vertices of the cycle graph and an edge denoted by uv is induced between the vertex u and vertex v. Then the edge labeling will be done as follows under the function $f^*$

$f^*$:E(T) –→ {0,1,2}

$f^*(e=uv) = 0$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 0 (mod3)$

$= 1$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 1 (mod3)$

$= 2$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 2 (mod3)$

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CASE 2: when n mod 6 ∈{3,4,5}

Vertex labeling will be done under the function *f*: V(T) −→ {0,1,2,...., n-1} as

$f(a_p) = p - 1;\ 1 \leq p \leq n-1$

Let $e_1$ be the first edge out of the n-edges in the $C_n$ graph. So when n mod 6 ∈{2,3} then the first edge $e_1$ will always be fixed to 0 i.e. $e_1 = 0$ and the remaining (n-1) edges will be mapped as follows:

$f^*$:E(T) - {$e_1$} −→ {0,1,2}

$f^*(e=uv) = 0$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 0 (\text{mod} 3)$

$= 1$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 1 (\text{mod} 3)$

$= 2$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 2 (\text{mod} 3)$

Hence, the star graph preserves 3-equitable Ceiling Average Labeling ∀ n ≥ 6

**Example:**

1) Star graph $S_8$ is a 3-equitable ceiling average labeling graph.

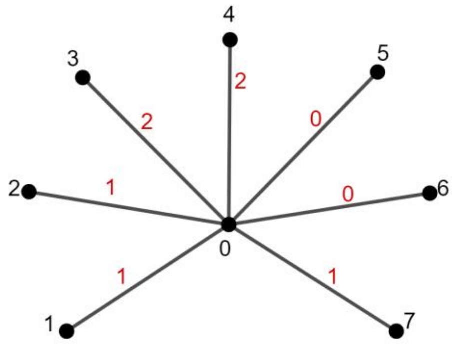


Figure 5: 3-equitable ceiling average labeling of $S_8$ graph

In Figure5, the red color numbers indicate the edge labeling.

1) Star graph $S_{15}$ is a 3-equitable ceiling average labeling graph.

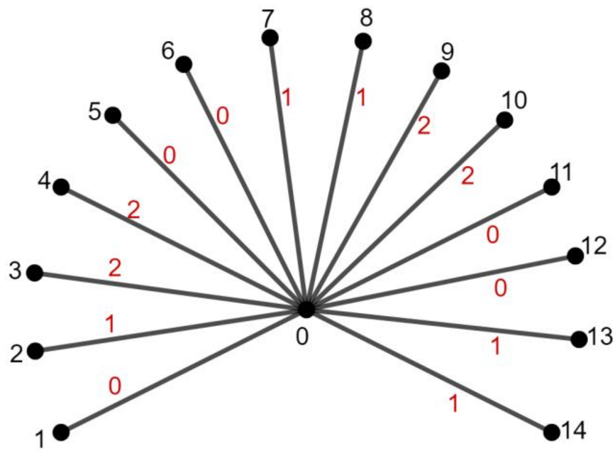


Figure 6: 3-equitable ceiling average labeling of $S_{15}$ graph

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In Figure6, the red color numbers indicate the edge labeling.

**THEOREM 2.4:** The wheel graph $W_n$ preserves 3-equitable Ceiling Average Labeling ∀ n ≥ 5.

**Proof:** Consider $a_1, a_2, a_3,..............., a_n$ the distinct vertices of a wheel graph $W_n$. The graph can be labeled according to a specific pattern involving the ceiling function and the mean of two consecutive vertex labels if:

- n mod 6 ∈{0,1,5} or
- n mod 6 ∈{2,3} then 0 is fixed at the edge of the wheel graph connecting the vertices labeled 0 and 1 or
- n mod 6 ∈{4} then 0 is fixed at the edge of the wheel graph connecting the vertices labeled 1 and 2.

CASE 1: when n mod 6 ∈{0,1,5}

Vertex labeling will be done under the function $f$: V(T) −→ {0,1,2,...., n-1} as

$f(a_p) = p - 1;\ 1 \le p \le n-1$

Let u and v are consecutive vertices of the cycle graph and an edge denoted by uv is induced between the vertex u and vertex v. Then the edge labeling will be done as follows under the function $f^*$

$f^*$:E(T) −→ {0,1,2}

$$f^*(e=uv) = 0 \text{ if } \lceil \frac{f(u)+f(v)}{2} \rceil \equiv 0(\text{mod}3)$$
$$= 1 \text{ if } \lceil \frac{f(u)+f(v)}{2} \rceil \equiv 1(\text{mod}3)$$
$$= 2 \text{ if } \lceil \frac{f(u)+f(v)}{2} \rceil \equiv 2(\text{mod}3)$$

CASE 2: when n mod 6 ∈{2,3}

Vertex labeling will be done under the function $f$: V(T) −→ {0,1,2,...., n-1} as

$f(a_p) = p - 1;\ 1 \le p \le n-1$

Let $e_1$ be the first edge out of the n-edges in the $W_n$ graph connecting the vertices labeled 0 and 1. So when n mod 6 ∈{2,3} then the first edge $e_1$ will always be fixed to 0 i.e. $e_1 = 0$ and the remaining (n-1) edges will be mapped as follows:

$f^*$:E(T) - {$e_1$} −→ {0,1,2}

$$f^*(e=uv) = 0 \text{ if } \lceil \frac{f(u)+f(v)}{2} \rceil \equiv 0(\text{mod}3)$$
$$= 1 \text{ if } \lceil \frac{f(u)+f(v)}{2} \rceil \equiv 1(\text{mod}3)$$
$$= 2 \text{ if } \lceil \frac{f(u)+f(v)}{2} \rceil \equiv 2(\text{mod}3)$$

CASE 3: when n mod 6 ∈{4}

Vertex labeling will be done under the function $f$: V(T) −→ {0,1,2,...., n-1} as

$f(a_p) = p - 1;\ 1 \le p \le n-1$

Let $e_1$ be the first edge out of the n-edges in the $W_n$ graph connecting the vertices labeled 1 and 2. So when n mod 6 ∈{4} then the first edge $e_1$ will always be fixed to 0 i.e. $e_1 = 0$ and the remaining (n-1) edges will be mapped as follows:

$f^*$:E(T) - {$e_1$} −→ {0,1,2}

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$f^*(e=uv) = 0$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 0(mod3)$

$= 1$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 1(mod3)$

$= 2$ if $\lceil \frac{f(u)+f(v)}{2} \rceil \equiv 2(mod3)$

Hence, the wheel graph preserves 3-equitable Ceiling Average Labeling ∀ n ≥ 5

**Example:**

1) Wheel graph $W_5$ is a 3-equitable ceiling average labeling graph.

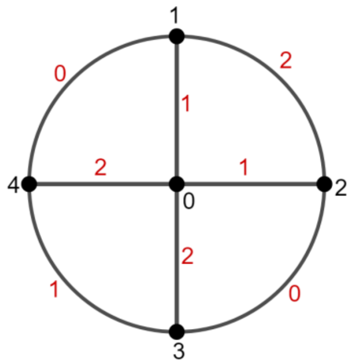


Figure 7: 3-equitable ceiling average labeling of $W_5$ graph

In Figure7, the red color numbers indicate the edge labeling.

2) Wheel graph $W_8$ is a 3-equitable ceiling average labeling graph.

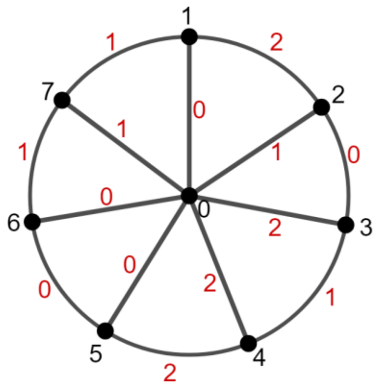


Figure 8: 3-equitable ceiling average labeling of $W_8$ graph

In Figure8, the red color numbers indicate the edge labeling.

3) Wheel graph $W_{16}$ is a 3-equitable ceiling average labeling graph.

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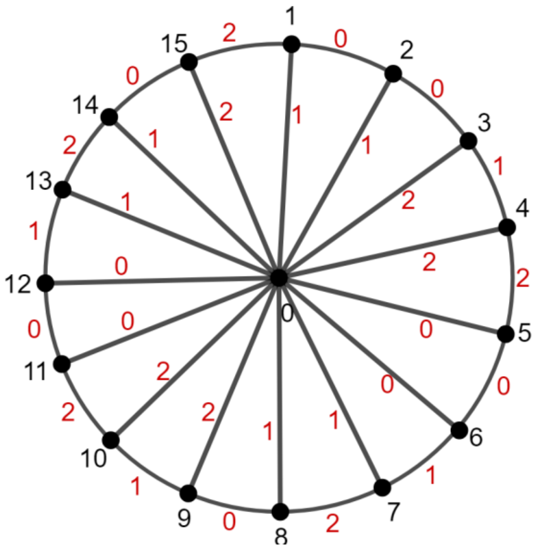

Figure 9: 3-equitable ceiling average labeling of $W_{16}$ graph

In Figure9, the red color numbers indicate the edge labeling.

## CONCLUDING REMARKS:

In conclusion, this paper has introduced a novel labeling pattern called 3-equitable Ceiling Average Labeling, which, to the best of our knowledge, has not been previously discussed in the existing literature. We have successfully applied this new labeling technique to several fundamental graph types, including cycle graphs, path graphs, star graphs, and wheel graphs, and have demonstrated its effectiveness and versatility across these structures. The results suggest that this labeling pattern offers a valuable addition to the family of graph labeling methods, with unique properties that warrant further investigation. The study has also highlighted the potential to extend this labeling pattern to more complex and diverse graph types. By applying it to other classes of graphs, such as bipartite graphs, complete graphs, or hypergraphs, we can further explore the theoretical foundations and practical applications of this labeling scheme. This opens up numerous opportunities for continued research and development, providing a new direction for future work in the field of graph theory and vertex labeling.

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