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Exploring the Structural Analysis of the Vectors Connectivity of Josephus Cube Interconnection Network: A Graph Theoretic Approach

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ABSTRACT

In this research, we studied the structural analysis of vectors of the Josephus Cube interconnection network. We explored the representation and properties of the graph model of the Josephus Cube and studied the relationship between the connectivity of nodes (processors). In this study, we have also presented the vertex-to-vertex and edge-to-edge connectivity matrix of the Josephus Cube using bitwise binary complements of the vector to find the complement of degree 4. We found the tautology between complements of different sets of nodes and chordal ring edge sets of the Josephus Cube.

Keywords: Interconnection Network, Graph Model, Vector, Connectivity Matrix, Binary Complement, Chordal Ring

1. Introduction

Graph theory is one of the most useful tools for modelling and analysing the vector connectivity of Josephus Cube (JC) interconnection networks. In a complex computational environment the importance of operations between processing elements (processors), such as information transfer is based on the structure of processing elements [1] [12]. The connectivity matrix of JC helps in reducing the number of relays between processing elements (units) such as processors, and defines the efficient communication between nodes/processors. The graph theory helps describe the best about these elements of a network e.g. diameter, degree, latency, and fault tolerance etc. [4] [10]. The common choice of problem of a communication network for parallel and distributed systems is to find a topology that can minimize the operations and linking of the communication structure [14] [15]. In association with the analysis of the structure, the JC interconnection network is a useful communication network for connecting parallel and distributed modules.

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Definition 1. An interconnection network is a technology of linking the number of processors for parallel and distributed computing systems [1] [2].

Definition 2. Josephus Cube is a type of interconnection network topology that follows the simple routing mechanism and is noted for its logarithmic diameter and excellent fault catholicity [3] [8].

Definition 3. It is a loosely coupled parallel processor that is based on a set of n-cube parallel processors, processors are made up of 2^r identical processors or vertices and there are 'r' connections (edges or links) that meet at each vertices. In other words, we can say that the Josephus cube is a topology of parallel processing systems, it may be a formal or structured interconnection network; it is also called an edge-augmented n-dimensional complete Josephus cube [3] [5].

Definition 4. It is like a recursive structure a similar to a hypercube of an interconnection network, where an n-cube is modeled by connecting two cubes that is $(n-1)$. The Josephus cube (JC) has a network diameter of $log_2 \frac{n}{2}$ where 'r' is the network size. The node degree of the Josephus cube is $[log_2 R] + 2 \; \forall \geq 1$. The total number of links 'r' is $r = 2^k, k \geq 1$ is given by $n \, log_2 \frac{n}{2} + \frac{3n}{4} = O(r \, log_2 \, r)$ [3].

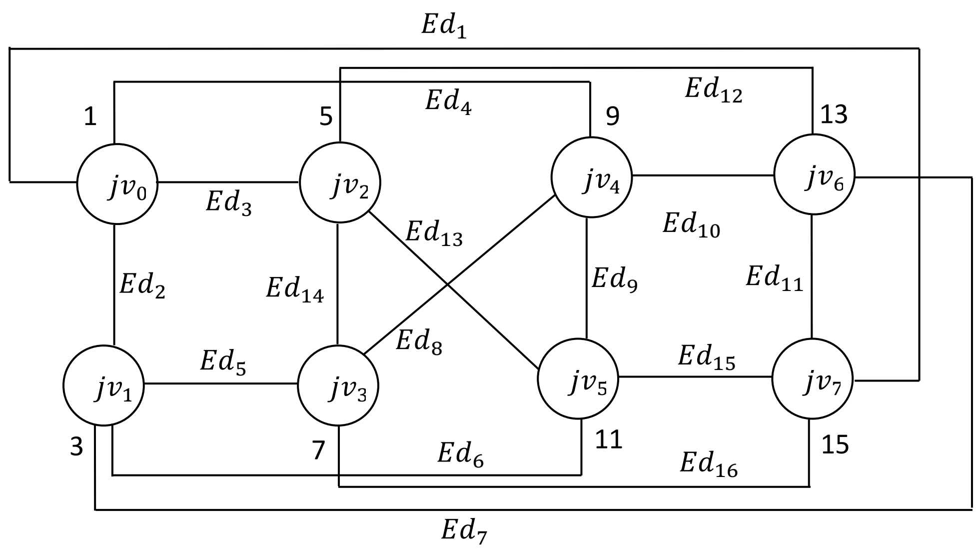

Fig. 1. Josephus cube $2^3 = 8$ nodes

Josephus cube graph in Fig. 1. is an undirected graph of $jv = 2^r$ (where jv is the number of nodes) vertices labelled from jv_0 to jv_7 $i.e. 2^r - 1$ Similarly, there are edges between two vertices as shown below. A Josephus cube has a r dimension, where r can be any positive integer including zero.

2. Representation of Graph Model Properties of Josephus Cube

We can explore or represent Josephus cube's graph model properties in three ways: the recursive/recursion using binary bit presentation method, the bitwise left shift and addition method, the bitwise XOR method and the difference method. All the methods are given in the next sections.

2.1. The recursive/recursion using binary bit representation

Josephus's cube is an undirected graph $G = (V, E)$ which is based on the Flavius Josephus problem. In this, we can represent $jv(i)$ based on the recursion and binary bit representation where the node is labelled using the function jv. Consider the following functions [3] [10]:

$jv(i) = 1$

$$jv(2i) = 2jv(i) - 1 \;\; where \;\; i \geq 1, and \; i \in \mathbb{Z}^+ \qquad (1)$$

$$jv(2i + 1) = 2jv(i) + 1 \;\; for \; i \geq 1, where \; i \in \mathbb{Z}^+ \qquad (2)$$

We can find similar results through the following:

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$jv(i) = 2(i - 2^n) + 1$ (3)

Where the highest power of 2 is equal to 0 is denoted by n.

$$i.e. b = 2$$
$$n = [log_b i]$$

Here in the above Josephus cube Fig.1, we consider the maximum $i = 8$ and
$2^n \leq i < 2^{n+1}$ (4)
Let's see an example in the below table as shown.

1	**2**	**3**	**4**
Decimal Numbers	2^r+< (< *is sequence followed by vertices*)	$jv(i)$ $2^1 < +1$	$jv(i)$ **Left Cyclic shift to the right**
1	1+0=1 (001)	2*0+1=1	1
2	2+0=2 (010)	2*0+1=1	01
3	2+1=3 (011)	2*1+1=3	011
4	4+0=4 (100)	2*0+1=1	001
5	4+1=5 (101)	2*1+1=3	011
6	4+2=6 (110)	2*2+1=5	101
7	4+3=7 (111)	2*3+1=7	111
8	8+0=8 (1000)	2*0+1=1	0001
9	8+1=9 (1001)	2*1+1=3	0011
10	8+2=10 (1010)	2*2+1=5	0101
11	8+3=11 (1011)	2*3+1=7	0111
12	8+4=12 (1100)	2*4+1=9	1001
13	8+5=13 (1101)	2*5+1=11	1011
14	8+6=14 (1110)	2*6+1=13	1101
15	8+7=15 (1111)	2*7+1=15	1111

Table 1 Recursive/recursion using binary bit representation and point cyclic shifting

We noticed the following topological properties based on the observations in the above Table 1:

1. If we work with different values of 'r' we will find a pattern here 1, 1,3,5,7,9 ... n.
2. If 'r' is a true power of 2, then the answer is always 1.
3. For every 'r' greater than the power of 2 the answer is incremented by 2.
4. If we see the binary representation in Table 1, the noticeable point is a one-bit left cyclic shift to the right, e.g. if 9 then the binary number is 1001 and it is $2^1 < +1 = 1001$ *and* $0011 = 3$, the same sequence node interconnection we can find from the equation (2).
5. A consistent recurring sequence node interconnection is obtained from Table 1 and if we compare it with Fig. 1, it shows that each node is connected with exactly four other nodes.

2.2. The bitwise left shift and addition method

In this we expressed i in the form of binary bits and shifted its bits to the left one place and then added 1 to the shifted bits but to get the pattern $j(i)$ should start from 0 in decimal and 0000 in binary, let consider the function:
$jv(i) = (i \ll 1) + 1$ (5)
Where $(i \ll 1)$ is the one-bit left shift, below in Table 2 shifted bits and procedures are shown as an example.

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Decimal Numbers	$J(i)$	$J(i << 1)$	$J(i << 1) + 1$	Node Connection Pattern
0	0000	0000	0001	1
1	0001	0010	0011	3
2	0010	0100	0101	5
3	0011	0110	0111	7
4	0100	1000	1001	9
5	0101	1010	1011	11
6	0110	1100	1101	13
7	0111	1110	1111	15

Table 2 Node Connection Pattern using bitwise left shift operation.

In Table 2, the node connection pattern is obtained for 1, 3, 5, 7, 9, 11, 13, and 15, as shown in Josephus Cube Fig. 1, where the sample of the sequence $j(i)$ for $0 \le i \le 7$ in decimal numbers and in binary representation. This connection pattern is obtained by half operations compared to recursive/recursion using the binary bit representation method where double of operations are required to obtain the same connection patterns.

2.3. The bitwise XOR method

To change the binary representation of the interconnection, we can use the XOR method rather than shifting.

$$jv(i) = (i \oplus 2^n) + 1 \tag{6}$$

Where $n = [log_2 i]$ and the most important bits are flips during XORing with 2^n. To make sure the rightmost bit is correctly moved, add 1.

2.4. The Difference Method

This method of representation of JC is based on addition, multiplication, and finally subtraction. Let's consider the following function:

$$x = (n+1) * (n+1) + 1 \tag{7}$$

$$p = x_i - x_{i+1} \tag{8}$$

Where $i \ge 1$ and initially $x_1 = 1$

Here, to get the pattern, we assume that i must be minimum $i = 1$ otherwise, 1 will be lost from the pattern.

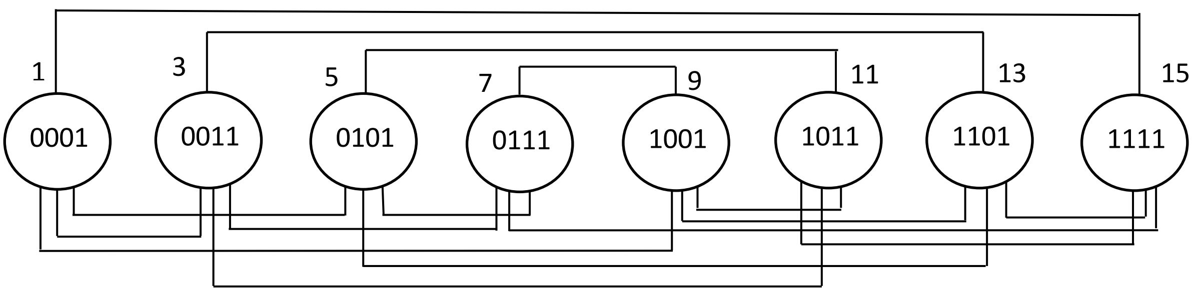

Fig. 2. Josephus cube the binary and decimal node connection pattern

The model illustrated in Fig. 2. will be derived using all the methods for the binary and decimal node interconnection patterns.

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3. Representation of the Connectivity Matrix of the Josephus Cube

Algorithms like Dijkstra's shortest path, Spanning Tree, PageRank, and graph traversal benefited and based on the connectivity matrix. The structure of JC's connectivity matrix allows parallel execution of these algorithms which improves efficiency in search, optimization, and machine learning applications. In terms of load balancing the connectivity matrix helps to manage computational loads between multiple processors in parallel computing environments [13] [14].

3.1. Vertex-to-Vertex Connectivity Matrix of the Josephus Cube

In Table 2, the connectivity matrix between the processors/nodes is shown, 1 if the processor is connected, otherwise 0 in the connectivity matrix.

	jv_0	jv_1	jv_2	jv_3	jv_4	jv_5	jv_6	jv_7
jv_0	0	1	1	0	1	0	0	1
jv_1	1	0	0	1	0	1	1	0
jv_2	1	0	0	1	0	1	1	0
jv_3	0	1	1	0	1	0	0	1
jv_4	1	0	0	1	0	1	1	0
jv_5	0	1	1	0	1	0	0	1
jv_6	0	1	1	0	1	0	0	1
jv_7	1	0	0	1	0	1	1	0

Table 3 vertex-to-vertex connectivity of Josephus Cube

In the vertex-to-vertex connectivity matrix, the node vectors show the complement and sequential relationship as well, where $[jv_0, jv_1]$, $[jv_2, jv_3]$, $[jv_4, jv_5]$, and $[jv_6, jv_7]$ are complementing each other. We also observed the following properties from the above connectivity matrix of the Josephus Cube:

1. The above Josephus cube matrix is $n * m$ matrix
2. Concurrent matrix
3. Tautology
4. Complement

A vector is a row and column in the connectivity matrix of the JC. Each row vector shows connectivity between four nodes/ processors. There are 8 nodes and each node has four edges, therefore the density of the matrix is $(\frac{8*4}{64} * 100)$ 50%. The matrix has the node vector from jv_0 to jv_7 and each nodes have its binary value shown in the connectivity matrix.

Lemma 1 The Matrix of Josephus cube is always n*m and in a Josephus cube matrix the row i is the complement of the row $(i + 1)$.

Proof: The adjacency matrix of the Josephus cube is shown in Table 2 and from the adjacency matrix of the Josephus cube matrix the first row $(i) = 01101001$ and $(i + 1) = 10010110$. Here in the first row (i) bit 0 is 1 at the row $i + 1$ similarly

1 is 0

1 is 0

0 is 1

1 is 0

0 is 1

0 is 1

1 is 0

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Which is the complement of each other.

Lemma 2 The connectivity of the Josephus Cube is the bitwise binary complements of node vectors and each node vector has a complement degree of 4.

Proof: Consider the vertex-to-vertex connectivity of Josephus Cube in the Table 3. We know that vector is a row and column in the connectivity matrix of the JC where by considering row and column vectors has the following bitwise complement of vectors:

(a) $\boldsymbol{jv_0} = 01101001$ The binary complements of the vector $\boldsymbol{jv_0}$ are $\boldsymbol{jv_1} = 10010110$, $\boldsymbol{jv}_2 = 10010110$, $\boldsymbol{jv_4} = 10010110$, $\boldsymbol{jv_7} = 10010110$
(b) $\boldsymbol{jv_1} = 10010110$ The binary complements of the vector $\boldsymbol{jv_1}$ are $\boldsymbol{jv_0} = 01101001$, $\boldsymbol{jv}_3 = 01101001$, $\boldsymbol{jv}_5 = 01101001$, $\boldsymbol{jv_6} = 01101001$
(c) $\boldsymbol{jv_2} = 10010110$ The binary complements of the vector $\boldsymbol{jv_2}$ are $\boldsymbol{jv_0} = 01101001$, $\boldsymbol{jv}_3 = 01101001$, $\boldsymbol{jv}_5 = 01101001$, $\boldsymbol{jv_6} = 01101001$
(d) $\boldsymbol{jv_3} = 01101001$ The binary complements of the vector $\boldsymbol{jv}_3$ are $\boldsymbol{jv_1} = 10010110$, $\boldsymbol{jv}_2 = 10010110$, $\boldsymbol{jv_4} = 10010110$, $\boldsymbol{jv_7} = 10010110$
(e) $\boldsymbol{jv_4} = 10010110$ The binary complements of the vector $\boldsymbol{jv}_4$ are $\boldsymbol{jv_0} = 01101001$, $\boldsymbol{jv}_3 = 01101001$, $\boldsymbol{jv}_5 = 01101001$, $\boldsymbol{jv_6} = 01101001$
(f) $\boldsymbol{jv_5} = 01101001$ The binary complements of the vector $\boldsymbol{jv_5}$ are $\boldsymbol{jv_1} = 10010110$, $\boldsymbol{jv}_2 = 10010110$, $\boldsymbol{jv_4} = 10010110$, $\boldsymbol{jv_7} = 10010110$
(g) $\boldsymbol{jv}_6 = 01101001$ The binary complements of the vector $\boldsymbol{jv_6}$ are $\boldsymbol{jv_1} = 10010110$, $\boldsymbol{jv}_2 = 10010110$, $\boldsymbol{jv_4} = 10010110$, $\boldsymbol{jv_7} = 10010110$
(h) $\boldsymbol{jv_7} = 10010110$ The binary complements of the vector $\boldsymbol{jv}_7$ are $\boldsymbol{jv_0} = 01101001$, $\boldsymbol{jv}_3 = 01101001$, $\boldsymbol{jv}_5 = 01101001$, $\boldsymbol{jv_6} = 01101001$

Hence, it is proved that each node vector has a binary complement with degree 4. Performing mathematical Logical 'OR' operations on given vectors we can get tautology, which validates and verifies the lemma.

Lemma 3: The connectivity matrix of the Josephus cube $JC(n)$, let jv_i be the i^{th} row vector, representing links of nodes i, then all nodes i holds the following property:

(a) Mutual connection symmetry
(b) Self-symmetry
(c) Complementary vector

Proof: Let consider the Fig.1, As we know that JC interconnected network is an undirected graph where edges are bidirectional, so that if $JC[i]$ is connected to $JC[j]$, then $JC[j]$ is connected to $JC[i]$ i.e. $JC[i][j] = JC[j][i] = 1$ that is nothing but it is Mutual connection symmetry. The property or structure of JC interconnection network where entire structure should be invariant. Technically holding same number of neighbours connected through edges by each node implies self-symmetry that is structurally identical and offers communication uniformity between nodes/processors.

Let $G = (V, E)$ be an undirected graph, and a graph is self-symmetric if two nodes $u, v \in V$.

A complementary vector is a vector where each vector has a minimum of at least one complement that is shown in Table 1 of the connectivity matrix and proved complement in lemma 1 and lemma 2.

Lemma 4 The set of row vectors $\{jv_0, jv_1\}$, $\{jv_2, jv_3\}$, $\{jv_4, jv_5\}$, $\{jv_6, jv_7\}$, $\{jv_0, jv_2\}$, $\{jv_0, jv_4\}$, $\{jv_0, jv_7\}$, $\{jv_1, jv_3\}$, $\{jv_1, jv_5\}$, $\{jv_1, jv_6\}$, $\{jv_2, jv_5\}$, $\{jv_2, jv_6\}$, $\{jv_3, jv_4\}$, $\{jv_3, jv_7\}$, $\{jv_4, jv_5\}$, $\{jv_4, jv_6\}$ and $\{jv_5, jv_7\}$ hold the binary relation and tautology in the vertex-to-vertex connectivity of the Josephus Cube.

Proof: A well-defined formula and rules of inference, where the logical statement may be either true or false, known as a tautology. The tautology is based on mathematical logical operations. Let us consider the vector values of Table 3 for a given set and perform logical 'OR' operation between them:

$$\{jv_0 \vee jv_1\} = (01101001 \vee 10010110) = 11111111$$

$$\{jv_2 \vee jv_3\} = (10010110 \vee 01101001) = 11111111$$

$$\{jv_4 \vee jv_5\} = (10010110 \vee 01101001) = 11111111$$

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$$\{jv_6 \vee jv_7\} = (01101001 \vee 10010110) = 11111111$$
$$\{jv_0 \vee jv_2\} = (01101001 \vee 10010110) = 11111111$$
$$\{jv_0 \vee jv_4\} = (01101001 \vee 10010110) = 11111111$$
$$\{jv_0 \vee jv_7\} = (01101001 \vee 10010110) = 11111111$$
$$\{jv_1 \vee jv_3\} = (10010110 \vee 01101001) = 11111111$$
$$\{jv_1 \vee jv_5\} = (10010110 \vee 01101001) = 11111111$$
$$\{jv_1 \vee jv_6\} = (10010110 \vee 01101001) = 11111111$$
$$\{jv_2 \vee jv_5\} = (10010110 \vee 01101001) = 11111111$$
$$\{jv_2 \vee jv_6\} = (10010110 \vee 01101001) = 11111111$$
$$\{jv_3 \vee jv_4\} = (01101001 \vee 10010110) = 11111111$$
$$\{jv_3 \vee jv_7\} = (01101001 \vee 10010110) = 11111111$$
$$\{jv_4 \vee jv_5\} = (10010110 \vee 01101001) = 11111111$$
$$\{jv_4 \vee jv_6\} = (10010110 \vee 01101001) = 11111111$$
$$\{jv_6 \vee jv_7\} = (01101001 \vee 10010110) = 11111111$$

There are total of 17 sets of vectors, after performing logical 'OR' operation between the given sets, each set holds tautological properties and binary relation of an interconnection network. The 'OR' operation gives the tautology that is useful in the communication network for error checking, data verification, and encryption.

Lemma 5 The binary complements hold symmetrical, dual, or inverse relationships of an interconnection network topology.

Proof: Since the binary complements maintain and ensure uniform connectivity between nodes/processors that it makes routing paths balanced. If any node fails, the binary complement of nodes will act as a backup in parallel and distributed execution. Let us consider the binary values of two vectors of the vertex-to-vertex connectivity matrix:

$$jv_0 = 01101001$$
$$jv_1 = 10010110$$

The vector jv_0 holds the complement binary values in the vector jv_1 and vice versa, that is also dual or inverse to each other, hence it is proved. The complements of nodes used for reducing the choke point of network traffic by managing the load effectively.

3.2. Edge-to-Edge connectivity Matrix and Chordal ring Sets of Josephus Cube

In the context of Josephus Cube Chordal Ring is a special type of an Interconnection network [9]. There are 16 edges in a given JC fig.1. Edge-to-edge connectivity matrix is shown in Table 4 below. It is used for the implementation of a local small network of interconnected processors.

	Ed_1	Ed_2	Ed_3	Ed_4	Ed_5	Ed_6	Ed_7	Ed_8	Ed_9	Ed_{10}	Ed_{11}	Ed_{12}	Ed_{13}	Ed_{14}	Ed_{15}	Ed_{16}
Ed_1	1	1	1	1	0	0	0	0	0	0	1	0	0	0	1	1
Ed_2	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
Ed_3	1	1	1	1	0	0	0	0	0	0	0	1	1	1	0	0
Ed_4	1	1	1	1	0	0	0	1	1	1	0	0	0	0	0	0
Ed_5	0	1	0	0	1	1	1	1	0	0	0	0	0	1	0	1
Ed_6	0	1	0	0	1	1	1	0	1	0	0	0	1	0	1	0
Ed_7	0	1	0	0	1	1	1	0	0	1	1	1	0	0	0	0

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Ed_8	0	0	0	1	1	0	0	1	1	1	0	0	0	1	0	1
Ed_9	0	0	0	1	0	1	0	1	1	1	0	0	1	0	1	0
Ed_{10}	0	0	0	1	0	0	1	1	1	1	1	1	0	0	0	0
Ed_{11}	1	0	0	0	0	0	1	0	0	1	1	1	0	0	1	1
Ed_{12}	0	0	1	0	0	0	1	0	0	1	1	1	1	1	0	0
Ed_{13}	0	0	1	0	0	1	0	0	1	0	0	1	1	1	1	0
Ed_{14}	0	0	1	0	1	0	0	1	0	0	0	1	1	1	0	1
Ed_{15}	1	0	0	0	0	1	1	0	1	0	1	0	1	0	1	1
Ed_{16}	1	0	0	0	1	0	0	1	0	0	1	0	0	1	1	1

Table 4 Edge-to-edge connectivity of a Josephus cube

The above connectivity matrix shows all possible connections between the edges of JC. The basic characters of JC lie in sets with the property of a chordal ring [11]. The number of bits in the row vector shows the connectivity with other edges through nodes/processors which is also the density of connectivity in the JC processors. Each row and column vector has connectivity with 7 other edges including self-edge. The density of the connectivity matrix is ($\frac{(16*7)}{16*16} * 100$) 43.75%. The binary value of row vectors and corresponding column vectors are the same e.g. $Ed_1 = 1111000000100011$.

Here, the Chordal ring sets have been given $Ed_1, Ed_2, Ed_3, Ed_4, Ed_5, Ed_{10}, Ed_{12}$ and Ed_{13} subsequently. Set plays an important role in the efficient and smooth routing process between edges and nodes.

$Ed_1 = \{Ed_1, Ed_2, Ed_3, Ed_4\}$

$Ed_2 = \{Ed_1, Ed_2, Ed_3, Ed_4\}$, $Ed_2 = \{Ed_2, Ed_3, Ed_4, Ed_5\}$, $Ed_2 = \{Ed_3, Ed_4, Ed_5, Ed_6\}$

$Ed_2 = \{Ed_4, Ed_5, Ed_6, Ed_7\}$

$Ed_3 = \{Ed_1, Ed_2, Ed_3, Ed_4\}$

$Ed_4 = \{Ed_1, Ed_2, Ed_3, Ed_4\}$

$Ed_5 = \{Ed_5, Ed_6, Ed_7, Ed_8\}$

$Ed_{10} = \{Ed_7, Ed_8, Ed_9, Ed_{10}\}$, $Ed_{10} = \{Ed_8, Ed_9, Ed_{10}, Ed_{11}\}$, $Ed_{10} = \{Ed_9, Ed_{10}, Ed_{11}, Ed_{12}\}$

$Ed_{12} = \{Ed_{10}, Ed_{11}, Ed_{12}, Ed_{13}\}$, $Ed_{12} = \{Ed_{11}, Ed_{12}, Ed_{13}, Ed_{14}\}$

$Ed_{13} = \{Ed_{12}, Ed_{13}, Ed_{14}, Ed_{15}\}$

The sets $Ed_1, Ed_2, Ed_3, Ed_4, Ed_5, Ed_{10}, Ed_{12}$ and Ed_{13} are Josephus sequence, where the set of connectivity links allows neighbour path discovery and routing. Set provides a structured model for links and parallel processing [6]. The set-theoretic and mathematical logical operations on the set of connectivity links give the common paths (for multiple routing) and alternative paths in the routing of the JC interconnection network [16].

Lemma 6: The chordal ring structure in a Josephus Cube of an interconnection network is defined by its edge-based sets (links) and is node exclusive:

(a) The participation of a node may be in multiple chordal rings
(b) But a chordal ring can be defined by the logical subset of edges, and not by a subset of disjoint nodes.

Proof: By definition 4 in JC, further links which are used for long-range connections can be derived through permutations, and they are cyclic and should not divide into exclusive sets. If $Ed_j \subset Ed$ is a set of chords (edges), then the cyclic structure of chords $Ed_{j_k} \subset Ed_j$ where k indexes the ring.

Edges in the JC can be part of many chordal edge sets, as shown in the above sets $Ed_1, Ed_2, Ed_3, Ed_4, Ed_5, Ed_{10}, Ed_{12}$, Ed_{13} and nodes can also occur in many edge-defined rings. Another way we can say that the edges Ed_j can be divided into subsets of $Ed_{j_1}, Ed_{j_2} \dots Ed_{j_n}$ where Ed_{j_k}creates a ring structure, hence the lemma holds.

4. Conclusion and Future Scope

In specific, we have presented methods and a graph model of the Josephus Cube, e.g. the bitwise left shift and addition method, the bitwise XOR method, and the Difference Method. We have also shown the connectivity matrix and proved complements between node vectors/processors. Complements of vector help to find backup and lost data during the routing and communication process. This study will allow us to investigate the reliability of systems in complex computational environments and will help to develop efficient, cost-effective algorithms and mechanisms as well. In the future, applying this study will help to embed various AI Techniques that allow parallel execution of algorithms, improving efficiency in search, optimization, and machine learning applications to enhance the processing and communication computation efficiently.

Declarations

- **Conflict of Interest:** The authors declare no conflict of interest.
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