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Research Article

Solving Mixed Voltera-Fredholm Integro-Differential Equation by Two Numerical Approximation Methods

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ABSTRACT

In this paper, we present and define second kind linear mixed voltera-fredholm-integro differential equation (MVF-IDE-SK) and use two iterative approximate methods like Successive approximation method (SAM) and Adomian decomposition method (ADM), to solve this problem and algorithms of procedure for methods. The numerical problems were solved by mention methods, the outcomes are compared with the exact solution. The study confirm that techniques are valid, and can be generally applied to solve MVF-IDE-SK. For additional illustration. The convergence and uniqueness solution of the problems are stated and proved. The comparison between approximate and exact solutions, the error analysis and the computational efficacy, respectively.

Keywords: Successive approximate, Adomian Decomposition, Mixed Voltera-Fredholm, Integro Differential Equations, Iterative Method.

1. Introduction

One of the important subjects in different area of applied mathematics are the theory and application of integral equations. There are several fields and problems which are belong to integral equation such as in mathematical physics, mathematics, many connected fields of engineering, mixed boundary value problems and contact problems in the theory elasticity led to integral equations (Mkhitaryan and Abdou, 1990) and (Wazwaz, 2011).

In this case integral equation is more suitable (Jerri, 1999). It was found that the sources and origins of integral equations in more details are in directed in (Estrada and Kanwal, 2012).

Our main interest subject, which is mixed linear Voltera-Fredholm type of integral equation arises from the mathematical modelling of the spatial temporal development of an epidemic, parabolic boundary value problem and in different biological and physical problems in electrical engineering in modeling of dynamic impulse system (Eidelman and Zhitarashu, 2012). (Sharif et al., 2025) investigates the existence and uniqueness of solutions for a nonlocal initial condition of the Caputo fractional Volterra-Fredholm integro-differential equation in a Banach space. We shall prove the existence and uniqueness of the results by using the Banach and Krasnoselskii fixed-point theorems. Differential equations with altered argument or differential equations of neutral type can be transformed to Voltera-Fredholm integral equations. The solution of Voltera-Fredholm integral equations can be obtained theoretically in (Abdou, 2003), at the same time in (Biazar and Eslami, 2010). The sensing of arithmetical methods takes an important place in solving, this type of equations. They had also been traced numerically by (Wang et al., 2014),(Ezzati and Najafalizadeh, 2012),(Ahmed, 2011) and (Sulaiman, 1999).

Where different methods are used to solve integro-differential equations arise quite frequently as mathematical models in diverse.

Consequently, numerous scientists have employed various techniques to find numerical solutions for integral equations. In (Abdou and Abd Al-Kader, 2005) Kader solved mixed type of Fredholm Volterra integral equation, Maleknejad solved a system of Volterra-Fredholm integral equations utilizing a computational method. In (Chen and Jiang, 2012) examined the approximate solution for mixed linear Volterra-Fredholm integral equations. In (Young and Martin, 2015), applied Newton's Raphson method to approximate solutions for non-linear equations. In (Hasan et al., 2016), discovered numerical solutions for linear systems of Volterra-Fredholm integral equations. Finally, in (Hasan, 2017) investigated systems of Volterra-Fredholm integral equations of the second type.

In (Hasan, 2016) utilized the Adomian decomposition method (ADM) to address the system of VFIE-2. In (Hasan et al., 2016),(Hasan, 2019) applied the Iterative Kernel Method to solve the two-Dimensional Linear Mixed Volterra-Fredholm Integral Equation of the Second Kind. To build upon these previous studies, the plan is to reformulate and apply ADM to tackle the TDIMVFIE in order to find its approximate solution.

In the field of applied mathematics, integral equations play crucial role in various applications. As highlighted by (Abdou, 2003), integral equations can arise in mathematical physics, and many connected fields of engineering. The volterra-Fredholm integral equations, in particular, have been subject of interest due to their applications in modeling dynamic impulse systems and biological and physical problems (Hassan and Hussein, 2025). The numerical methods for solving these equations have been extensively studied, with technique (Ahmed, 2011), and the CAS wavelets method (Ezzati and Najafalizadeh, 2012) providing valuable tools for researchers. The Adomian decomposition method (ADM) has also been applied to solve integral equations, offering a powerful approach to finding approximate solutions (Jerri, 1999), (Rahman, 2007). Furthermore, the iterative methods such as the Successive Approximation Method (SAM) have been shown to be effective in solving Volterra-Fredholm integral equations, as demonstrated by Sulaiman (Sulaiman, 1999). These methods have been further developed and applied to mixed Volterra-Fredholm integral equations of the second kind, contributing to the advancement of numerical analysis in this area (Wazwaz, 2011).

For extending these works, reformulating and using SAM and ADM for treating MVF-IDE-SK for finding the approximate solution of it.

2. Fundamentals of Volterra-Fredholm Integral -Differential Equation

This section presents some basic definition and theorem related to the work.

Definition 2.1: The linear mixed VF-IDE-SK is defined as follows:

$$z_n^{(m)}(k) = h(k) + \lambda \int_a^k \int_a^b H(k,t) z_{n-1}(t) dt\, dk. \quad (1)$$

With initial conditions

$$z(a) = a_0, \ z'(a) = a_1, \ldots, \ z^{(m-1)}(a) = a_{(m-1)}. \quad (2)$$

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Where $k \in [a, b]$ and $h(k)$ *is* a continuous function defined on the interval $[a, d]$. The functions $H(k,t)$ is continuous functions on $\{(k,t), a \leq t \leq k \leq b\}$ where λ is constants, while $z(t)$ is the unidentified function to be determined(Wazwaz, 2011).

Definition 2.2: The equation of the form

$$r(k)z(k) = h(k) + \lambda \int_a^k G(k,\ t)z(t)dt + \lambda^* \int_a^b G^*(k,\ t)z(t)dt. \qquad (3)$$

Termed linear Volterra-Fredholm integral equation, where $r(k)$ and $h(k)$ are identified continuous functions on [a, b], while $z(k)$ is unknown function. $G(k,\ t)$ and $G^*(k,\ t)$ are termed kernels of the integral equation and they are continuous functions on $\{(k,\ t): a \leq\ t \leq k \leq b\}$, λ and λ^* are called scalar parameter (Hasan, 2017).

Definition 2.3: Let (X, d) be a non-empty complete metric space, and let $T: X \to X$ be a contraction mapping, meaning there exist a constant $k \in [0,1)$ such that $d\big(T(x), T(y)\big) \leq k \,.\, d(x, y)$ for all $x, y \in X$. Then:

T has a unique fixed point $x^* \in X$, i.e., $T(x^*) = x^*$.For any initial point $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ converges to x^*(Jachymski et al., 2024).

Theorem 1: Let $\boldsymbol{f(k)}$ be a continuous function on $[\boldsymbol{a}, \boldsymbol{b}]$ then $\int_a^b \boldsymbol{f(t)dt}$ is bounded. **(Rahman, 2007)**

Theorem 2:(Fundamental theorem in Calculus)

If f is continuous at every point in [a, b] and F is any antiderivative of f on $[a, b]$, then.

$$\int_a^b f(t)dt = F(b) - F(a)$$

Theorem 3: (Convergence and Uniqueness)

Suppose $h(k)$, is a continuous function on the interval $[a, b]$ and the kernel function $H(k, t)$ is continuous on the region$\{(k, t) | a \leq t \leq b, a \leq k \leq b\}$. Let λ be a constant. If there exist constants M_1 such that $|\boldsymbol{H(k,t)}| \leq \boldsymbol{M_1}$ for all $\boldsymbol{k, t} \in [\boldsymbol{a}, \boldsymbol{b}]$.

1- $|\boldsymbol{\lambda}| \boldsymbol{M_1 (b-a)(k-a)} < \mathbf{1}$.
Then the equation:

$$\boldsymbol{z_n'(k) = h(k) + \lambda \int_a^k \int_a^b H\,(k,t) z_{n-1}\,(t)dt\,dk}.$$

Has a unique solution $\boldsymbol{z(k)}$ on the interval $[\boldsymbol{a}, \boldsymbol{b}]$.

Proof:

To prove this theorem, we will use the Banach fixed point theorem (also known as the contraction mapping theorem). First, we transform the equation into a fixed-point problem.

1. Define the Banach space

Consider the Banach space $C([a, b])$ with the norm:

$$\|z\| = \max_{k \in [a,b]} |z(k)|.$$

Here, $C([a, b])$ is the set of all continuous functions on the interval $[a, b]$.

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2. Construct the mapping T Define the mapping $T: C([a,b]) \to C([a,b])$ such that:
   $T\big(z'(k)\big) = h(k) + \lambda \int_a^k \int_a^b H(k,t)z(t)dt\,dk.$ (5)
   We need to show that T is a contraction mapping.
3. Verify contraction

For any $z_1, z_2 \in C([a,b])$, we compute:

$$|z_1(k) - z_2(k)| = |z_1(k) - z(a) + z(a) - z_2(k)| = |z_1(k) - z(a) - (z_2(k) - z(a))|.$$

By using fundamental theorem of calculus

$$\left|\int_a^k z_1'(t)dt - \int_a^k z_2'(t)dt\right| = \|\mathrm{T}(\mathrm{z_1}'(\mathrm{k})) - \mathrm{T}(z_2'(k))\|$$

$$= \left\|\lambda \int_a^k \int_a^b H(k,t)[z_1(t) - z_2(t)]dt\,dk\right\|$$

Using the properties of absolute values and integrals, and the boundedness of the kernel function:

$$\left|\int_a^k \int_a^b H(k,t)[z_1(t) - z_2(t)]dt\,dk\right| \le \int_a^k \int_a^b |H(k,t)|\,|z_1(t) - z_2(t)|dt$$

Since $|H(k,t)| \le M_1$ and $|z_1(t) - z_2(t)| \le \|z_1 - z_2\|$, we have:

$$\int_a^k \int_a^b |H(k,t)|\,|z_1(t) - z_2(t)|dt \le M_1.(b-a)\|z_1 - z_2\|.(k-a).$$

Thus:

$$\|T(z_1(k)) - T(z_2(k))\| \le |\lambda| M_1.(k-a)(b-a)\|z_1 - z_2\|.$$

Given the condition $|\lambda| M_1.(b-a)(k-a) < 1$, the mapping T is contraction.

To show that the equation (5) has a unique solution. By contradiction, suppose that there exist another solution U^*such that $U^* \neq Z^*$ which can satisfy $Z^* = T(Z^*)$ and $U^* = T(U^*)$, then U^* *and* Z^* are fixed points. Now $|\,U^* - Z^*| = |T(U^*) - T(Z^*)| \le L|\,U^* - Z^*|$, because T is Lipchitz function. Then we get $|\,U^* - Z^*| \le L|\,U^* - Z^*|$ hence $L \ge 1$which is contradiction to the definition of the contraction mapping because $L < 1$. Since T is contraction mapping on the region R we get $U^* = Z^*$ therefor the equation of MVF-IDE-SK has a unique solution.

4.Proposing SAM for solving linear mixed VF-IDE-SK

Consider the linear mixed voltera-fredholm-integro differential equation of the second kind.

Let $z(k) = \sum_{n=0}^{\infty} z_n(k)$ is approximate solution of MVFIDE and substitute in equation (1) we get

$$\frac{d^m}{dk^m}\left(\sum_{n=0}^{\infty} z_n(k)\right) = h(k) + \lambda \int_a^k \int_a^b H(k,t)\left(\sum_{n=0}^{\infty} z_n(k)\right) dt\,dk. \quad (6)$$

We go on after some abbreviations we get.

If $m = 1$ *and* $z_0(k) = 0$

$$\mathrm{z}_1'(k) = h(k) + \lambda \int_a^k \int_a^b H(k,t)z_0(t)dt\,dk. \quad (7)$$

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Integrating equation (6) from 0 to x we get $z_1(k)$

Then, $z_1(k)$ substituted again in equation (1) to get the second approximation $z_2(k)$,

$$z_2'(k) = h(k) + \lambda \int_a^k \int_a^b H(k,t) z_1(t) dt\, dk. \quad (8)$$

Integrating from 0 to x $z_2'(k)$ to get $z_2(k)$.

In general, this process can be continued to get the nth approximation

$$z_n'(k) = h(k) + \lambda \int_a^k \int_a^b H(k,t) z_{n-1}(t) dt\, dk. \quad (9)$$

Then determine whether $z_n(k)$ approaches the solution $z(k)$ as n increases (Rahman, 2007). It turns out that if $h(k)$ is continuous on $a \leq k \leq b$ and $H(k,t)$ is also continuous on {(k, t), $a \leq t \leq k \leq b$}, then it can be proved that the sequence $z_n(k)$ will converge to the solution $z(k)$ of (1). This will be done by proving the following:

Integrating $z_n^{\mathrm{m}}(k)$ m times to get $z_n(k)$.

i. The sequence $\{z_n(k)\}$ converges to a limit $z(k)$, this means $\lim_{n\to\infty} z_n(k) = z(k), a \leq k \leq b$
ii. $z(k)$ is a solution of (1) on $a \leq k \leq b$.
iii. The solution $z(k)$ of (1) is unique.

For the zero approximation $z_0(k)$, as mentioned before, any real-valued function can be used. The most common function used for $z_0(k)$ is $0, 1, or\ x$ (Rahman, 2007).

5. The SAM Algorithm for VF-IDE-SK

Input: a, b, k, n, m and $z_0(k)$.

Step 1: Assume $z(k) = \sum_{n=1}^{\infty} z_n(k)$. (*)

Step 2: Substitute equation * in equation (1) for $n = 1\ to\ \infty$.

Step 3: Taking integration of equation (8) from $0\ to\ x$ $z_1^{(m)}(k)$ over the interval $[a, k]$ with respect to k.

Step 4: By using the initial conditions obtained by equation (8), to get $z_1(k)$.

Step 5: Integrate $z_l^{(m)}(k)$ as mentioned in step 3, to get $z_l(k)$. We see that $z_l(k) \rightarrow z(k)$ as $l \rightarrow \infty$.

Output: the results of approximate solution.

6. Adomian Decomposition Method.

George Adomian created and presented the Adomian decomposition method (ADM) in 1981. In both linear and non-linear mathematical situations, this approach has been applied to solve differential and integral equations. This method's primary benefit is its straightforward application to a wide range of differential and integral equations, whether they are linear or non-linear, with variable or constant coefficients. The method's ability to significantly minimize the amount of computing labor while preserving the high accuracy of the approximate answer is another significant benefit. This

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method breaks down every equation's unknown function z(k) into a sum of an endless number of parts that are determined by the decomposition series. The series converges to an exact solution very quickly (Nhawu et al., 2016).

Recently, this technique has been applied for solving LSVIDE-SK and the linear system of VF-IE-SK by (Saeed, 2006) and (Hasan, 2017) respectively.

6.1 Description of the Method.

The decomposition method consists of representing k in equation (5) as a series $z(k) = \sum_{n=0}^{\infty} z_n(k)$ where the terms k_n are calculated by the following algorithm:

$z_0(\text{k}) = h(k)$, If $m = 1$.

Let $z(k) = \sum_{n=0}^{\infty} z_n(k)$ is approximate solution of MVFIDE, and substitute in equation (1) we get

$$\frac{d^m}{dk^m}\left(\sum_{n=0}^{\infty} z_n(k)\right) = h(k) + \lambda \int_a^k \int_a^b H\,(k,t)\left(\sum_{n=0}^{\infty} z_n(k)\right) dt\, dk. \quad (9)$$

We go on after some abbreviations we get

$$\text{z}_1'(k) = h(k) + \lambda \int_a^k \int_a^b H\,(k,t) z_0(t) dt\, dk. \quad (10)$$

Integrating equation (9) from 0 to x we get $z_1(k)$.

Then, $z_1(k)$ substituted again in equation (1) to get the second approximation $z_2(k)$,

$$z_2'(k) = h(k) + \lambda \int_a^k \int_a^b H\,(k,t) z_1(t) dt\, dk. \quad (11)$$

Integrating from 0 to x $z_2'(k)$ to get $z_2(k)$.

In general, this process can be continued to get the nth approximation

$$z_n'(k) = h(k) + \lambda \int_a^k \int_a^b H\,(k,t) z_{n-1}(t) dt\, dk. \quad (12)$$

Then determine whether $z_n(k)$ approaches the solution $z(k)$ as n increases (Rahman, 2007). It turns out that if $h(k)$ is continuous on $a \le k \le b$ and $H(k,t)$ is also continuous on {(k, t), $a \le t \le k \le b$}, then it can be proved that the sequence $z_n(k)$ will converge to the solution $z(k)$ of (1). This will be done by proving the following:

Integrating $z_n^{\text{m}}(k)$ m times to get $z_n(k)$

i. The sequence $\{z_n(k)\}$ converges to a limit $z(k)$, this means $\lim_{n\to\infty} z_n(k) = z(k), a \le k \le b$
ii. $z(k)$ is a solution of (1) on $a \le k \le b$.
iii. The solution $z(k)$ of (1) is unique.

These results can be proved as the corresponding proof for ordinary differential equations (Coddington, 2012).

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For the zero approximation $z_0(k)$, as mentioned before, any real-valued function can be used. The most common function used for $z_0(k)$ is $0, 1, or\ x$ (Rahman, 2007).

$$z_1'(k) = h(k) + \lambda \int_a^k \int_a^b H(k,t) z_0(t) dt\, dk. \qquad (13)$$

Integrating equation (6) from 0 to x we get $z_1(k)$ and use initial conditions

Then, this $z_1(k)$ is substituted again in equation (1) to get the second approximation $z_2(k)$,

$$z_2'(k) = h(k) + \lambda \int_a^k \int_a^b H(k,t) z_1(t) dt\, dk. \qquad (14)$$

Integrating from 0 to x $z_2'(k)$ to get $z_2(k)$.

In general, this process can be continued to get the nth approximation

$$z_n'(k) = h(k) + \lambda \int_a^k \int_a^b H(k,t) z_{n-1}(t) dt\, dk \qquad (15)$$

$z_1(\mathrm{k}) = B_0,$

$z_2(\mathrm{k}) = B_1,$

$\vdots$

$z_n(\mathrm{k}) = B_{n-1},$

Where B_n are called Adomian's polynomials, which are defined by:

$$B_n = \int_a^k \int_a^b H(k,t) [\textstyle\sum_{t=0}^{\infty} z_t]_{\lambda=0} dt\, dk \quad n = 0, 1, 2, \dots \qquad (16)$$

According to the above algorithm, we see that the solution z_j can be determined by the calculation of B_n, and we have

$z(k) = \sum_{n=0}^{\infty} z_n = h(s) + \sum_{n=0}^{\infty} B_n$ (Saeed, 2006).

6.2 The Application of ADM on Linear VF-IDE-SK.

In this section, we reformulate ADM for the first time to solve the linear VF-IDE-SK as followers:

Recall equation (1)

$$z^{(m)}(k) = g(k) + \lambda \int_a^k \int_a^b H(k,t) z(t) dt\, dk \qquad (17)$$

$m = 1, 2, 3, \dots$

With the following initial conditions:

$$z(a) = a_0,\ z'(a) = a_1, \dots,\ z^{(m-1)}(a) = a_{(m-1)}.$$

Assume we have the following assumption:

$$z_0^{(m)}(k) = h(k).$$

After the integration of above equation m-times with respect to s, over the interval $[a, s]$ and using the initial conditions given in (2), we get the initial solution

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$$z_0(k) = H(k).$$

Using equation, we get

$$z_1^{(m)}(k) = \lambda \int_a^k \int_a^b H\,(k,t)\, \frac{d^0}{d\lambda^0} [\textstyle\sum_{t=0}^{\infty} \lambda^t z_t]_{\lambda=0} dt\, dk. \quad (18)$$

After integration and using the initial conditions as mentioned before, we obtain $z_1(k)$ which is equal to B_0.

After l iterations, the ADM gives the following:

$$z_l^{(m)}(k) = \frac{\lambda}{(l-1)!} \int_a^k \int_a^b H\,(k,t)\, \frac{d^{(l-1)}}{d\lambda^{(l-1)}} [\textstyle\sum_{t=0}^{\infty} \lambda^t z_t]_{\lambda=0} dt\, dk.$$

After solving the above equation with the help of the initial conditions given in (2) for finding $z_l(k)$ which is equal to B_{l-1} , we get $l - term$ approximation $Y_l(k)$ to the solution $v(s)$ where:

$$Y_l(k) = \textstyle\sum_{n=0}^{l} z_n(k) = H(k) + \sum_{n=0}^{l-1} B_n\ . \quad (19)$$

With $\lim_{l\to\infty} Y_l = z(k)$.

6.3 The ADM Algorithm for VF-IDE-SK

Input: a, b, k, n, m and $z_0(k)$.

Step 1: Integrate $z_0^{(m)}(k) = h(k)$ over the interval $[a, k]$ with respect to s,

and using the initial conditions given in (2) to get $z_0(k)$.

Step 2: Use equation (13) to obtain equation (15).

Step 3: Integrate equation (15) as in step one.

Step 4: After l-iteration we obtain $Y_l(k)$. See equation (16).

Output: the results of approximate solution.

7. Numerical Examples

In this paper we solve some numerical examples

Example 1:

To solve the following linear Integro differential equation of the second kind with initial condition $z(0) = 1$

$$z'(k) = e^k - k + \int_0^k \int_0^1 t\, z(t) dt\, dk$$

It is clear that $f(k) = e^k - k$, $H(x,t) = t$

With the exact solution, $z(k) = e^k$, using SAM and ADM.

(i) Using SAM: The solution with iteration four is

$$z_4(k) = exp(k) - \frac{k^2}{1024}.$$

The solution with eight iterations is

$$z_8(k) = exp(k) - \frac{k^2}{4194304}.$$

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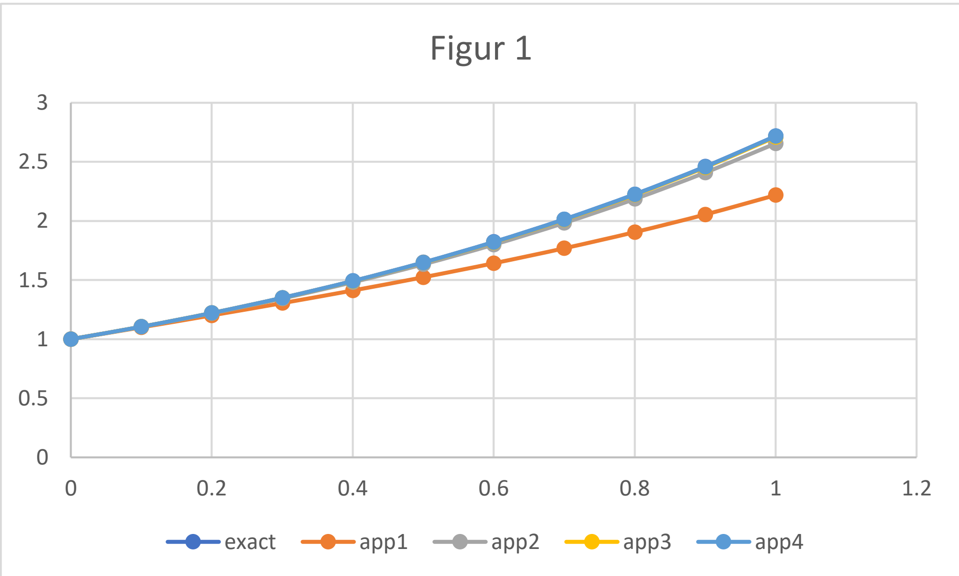
The solution with twelve iterations is

$$z_{12}(k) = exp(k) - \frac{k^2}{17179869184}.$$

Note: The errors approach to zero after 12 iterations.

Table 1 The fallouts for the approximate solution *z(k)* of Example 1,**using SAM.**

k	Exact Solution	Approximate Solution		
		4 iterations	8 iterations	12 iterations
0	1	1	1	1
.1	1.105170918	1.105161152	1.105170916	1.105170918
.2	1.221402758	1.221363695	1.221402749	1.221402758
.3	1.349858807	1.349770916	1.349858786	1.349858807
.4	1.491824697	1.491668447	1.491824659	1.491824697
.5	1.648721270	1.648477130	1.648721211	1.648721270
.6	1.822118800	1.821767237	1.822118715	1.822118800
.7	2.013752707	2.013274191	2.013752591	2.013752707
.8	2.225540928	2.224915928	2.225540776	2.225540928
.9	2.459603111	2.458812095	2.459602918	2.459603111
1	2.718281828	2.717305265	2.718281590	2.718281828
L.S.E		**2.41594E-06**	**1.44001E-13**	**8.58E-21**



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(ii) Using ADM: The solution with iteration four is

$$z_4(k) = exp(k) - \frac{k^2}{3072}.$$

The solution with eight iterations is

$$z_8(k) = exp(k) - \frac{k^2}{12582912}.$$

The solution with twelve iterations is

$$z_{12}(k) = exp(k) - \frac{k^2}{51539607552}.$$

Note: The errors approach to zero after 12 iterations.

Table 2 The fallouts for the approximate solution *z(k)* of Example 1 **using ADM.**

k	Exact Solution	Approximate Solution		
		4 iterations	8 iterations	12 iterations
0	1	1	1	1
0.1	1.105170918	1.105167663	1.105170917	1.105170918
0.2	1.221402758	1.221389737	1.221402755	1.221402758
0.3	1.349858808	1.349829511	1.349858800	1.349858808
0.4	1.491824698	1.491772614	1.491824685	1.491824698
0.5	1.648721271	1.648639890	1.648721251	1.648721271
0.6	1.822118800	1.822001613	1.822118772	1.822118800
0.7	2.013752707	2.013593202	2.013752669	2.013752707
0.8	2.225540928	2.225332595	2.225540878	2.225540928
0.9	2.459603111	2.459339439	2.459603047	2.459603111
1	2.718281828	2.717956308	2.718281749	2.718281828
L.S.E		**2.68438E-07**	**2.72087628e-14**	**9.53678E-22**

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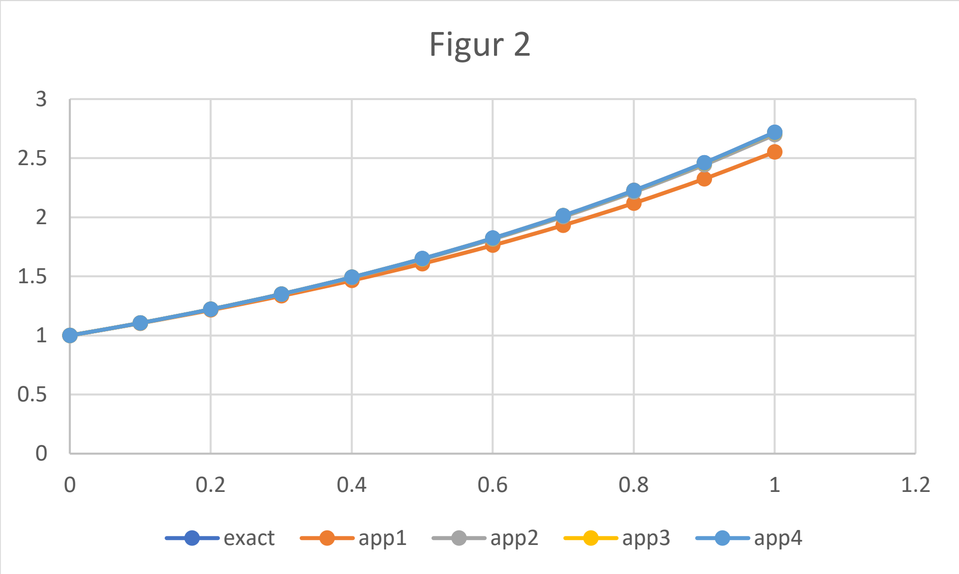

Example 2: To solve the following linear Integro differential equation of the second kind with initial condition $z(0) = 0$

$$z'(k) = \cos(k) - \frac{k^2}{2} + \int_0^k \int_0^1 st\, z(t)dt\, dk$$

It is clear that $f(k) = \cos(k) - \frac{k^2}{2}$, $H(x,t) = st$

With the exact solution, $z(k) = \sin(k)$, using S.A.M and A.D.M.

(i) Using S.A.M.: The solution with iteration four is

$$z_4(k) = sin(k) - \frac{k^3 * pi^{15}}{5308416000}$$

The solution with eight iterations is

$$z_8(k) = sin(k) - \frac{k^3 * pi^{35}}{450868486864896000000}$$

The solution with twelve iterations is

$$z_{12}(k) = sin(k) - \frac{k^3 * pi^{55}}{38294359833110460235776000000000}$$

Note: The errors approach to zero after 12 iterations.

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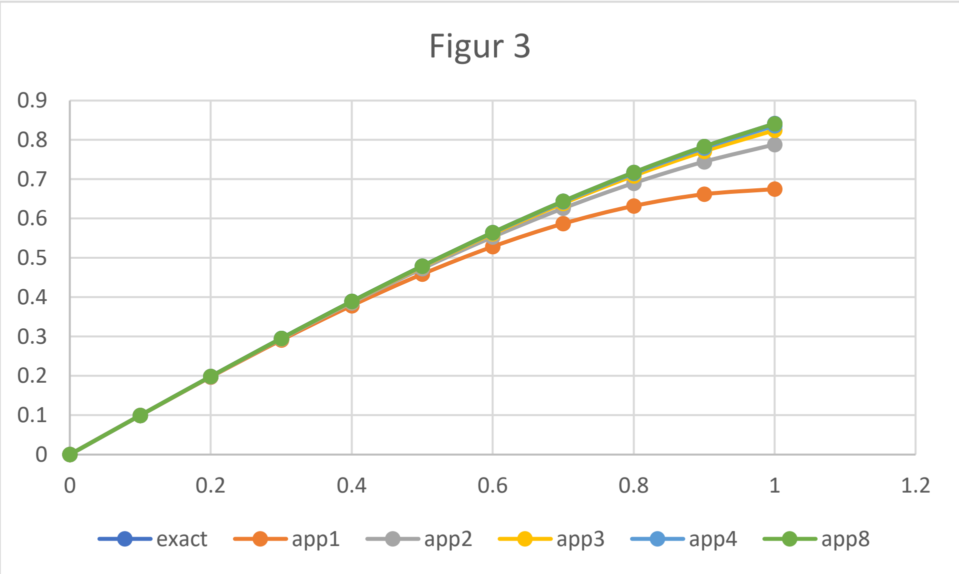
Table 3 The fallouts for the approximate solution *z(k)* of Example 2,using SAM.

K	Exact Solution	4 iterations	8 iterations	12 iterations
0	1	0	0	0
0.1	0.099833417	0.099828018	0.099833361	0.099833417
0.2	0.198669331	0.198626142	0.198668885	0.198669331
0.3	0.295520207	0.295374444	0.295518702	0.295520207
0.4	0.389418342	0.389072830	0.389414775	0.389418342
0.5	0.479425539	0.478750711	0.479418571	0.479425539
0.6	0.564642473	0.563476370	0.564630433	0.564642473
0.7	0.644217687	0.642365959	0.644198567	0.644217687
0.8	0.717356091	0.714591995	0.71732755	0.717356091
0.9	0.783326909	0.779391312	0.783286272	0.0.78328627
1	0.841470985	0.836072360	0.841415241	0.841470985
L.S.E		**5.76609E-05**	**6.14761E-09**	**6.55438E-13**



(ii) Using ADM: The solution with iteration four is

$$z_4(k) = sin(k) - \frac{k^3 * pi^{15}}{2654208000} + \frac{k^3 * pi^{16}}{10616832000} - \frac{k^3 * pi^{19}}{679477248000}.$$

The solution with eight iterations is

$$z_8(k) = sin(k) - \frac{k^3 * pi^{35}}{2254342434324480000000} + \frac{k^3 * pi^{36}}{9017369737297920000000} - \frac{k^3 * pi^{39}}{577111663187066880000000}.$$

The solution with twenty iterations is

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$$z_{20}(k) = sin(t) - \frac{t^3 * pi^{95}}{13812576058731491615588346216974175436800000000000000000000} + \frac{t^3 * pi^{96}}{55250304234925966462353384867896701747200000000000000000000} - \frac{t^3 * pi^{99}}{35360194710352618535906166315453889118208000000000000000000000}.$$

Note: The errors approach to zero after 12 iterations

Table 4 The fallouts for the approximate solution *z(k)* of Example 2, by using ADM

k	ct Solution	roximate Solution		
		erations	erations	terations
		0		
	0.099833417	0.099826991	99833350	99833416
	0.198669331	0.198617927	0.198668800	98669325
	0.295520207	0.295346718	0.295518415	95520188
	0.389418342	0.389072830	0.389414096	89418298
	0.479425539	0.478750710	0.479417245	79425453
	0.564642473	0.563254563	0.564628142	64642325
	0.644217687	0.642013738	0.644194930	44217452
	0.717356091	0.714066230	0.717322121	17355740
	0.783326914	0.778642714	0.783278543	83326410
	0.841470985	0.835045476	0.841404638	41470299
E		**8.16827E-05**	**8.70874E-09**	**9.28497E-13**

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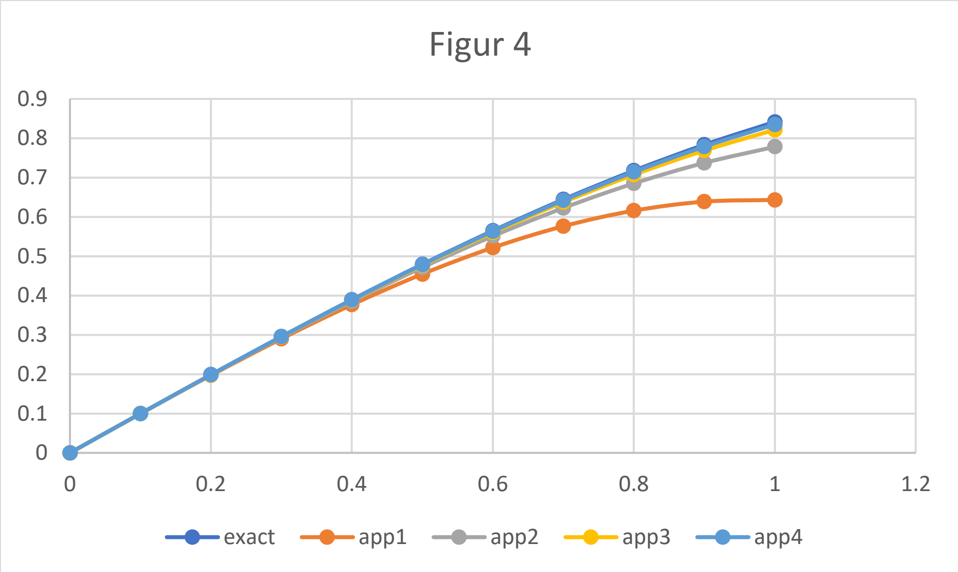

8. Analysis and Discussion

The study evaluates the SAM and ADM for solving linear VF-IDE-SK. Both methods exhibit strong convergence properties, whith approximate solution closly matching the exact solutions after a small number of iterations. Errors decrease rapidly, approaching near-zero values after 12 iterations for example 1 and 2, highlighting their accuracy and reliability. SAM and ADM are computationally efficient, featuring straightforward algorithms that are easy to implement, particularly with MATLAB for computation and visualization. SAM relies on Successive Aproximation and integrations, which can be intensive for higher-order equations, while ADM simplifies computations by decomposing the unknown function into a series. The choice between the two depends on the problem context and the balance between computational effort and accrancy. The application of these methods to numerical examples confirms their validity and practical effectiveness. Results align closely with exact solutions, and the extremely small absolute errors further emphasize their efficiency. While SAM and ADM yield accurate results, their distinct characteristics make them suitable for different problem scenarios. Overall, the study underscores the robustness of these methods as reliable tools for solving complex integro-differential equations.

9. Conclusion

In conclusion, the study successfully demonstrates the effectiveness of the Successive Approximation Method (SAM) and the Adomian Decomposition method (ADM) in solving **VF-IDE-SK.** Both methods provide accurate approximate solutions that closely match the exact solutions,with errors approaching zero after a resonable number of iterations. The algorithms are computationally efficient and easy to implement, making them valuable tools for numerical analysis in applied mathematics. The results of this study confirm that SAM and ADM are valid and reliable methods for solving **VF-IDE-SK**, and their application can be extended to various fields, including mathematical physics, engineering, and biological modeling. Figures 1,2,3 and 4 show the convergence rate of both methods the comparison of solutions for different iteration numbers the error reduction over iterations and the application to real-world problems respectively. The use MATLAB for numerical computations and error analysis further enhances the practical utility of these methods.

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