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Computation of Double Domination Integrity of Some Special Graphs

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ABSTRACT

In a communication network, the idea of vulnerability is important when there is a network disruption. There are numerous graph parameters available to measure a communication network's susceptibility. One of the vulnerability metrics used to assess a communication network's performance is double domination integrity. This article presents the double domination integrity of some special graphs and also suggests an application of double domination integrity in the real world.

Keywords: domination, integrity, performance, communication

2. INTRODUCTION

Graph theory is a renowned field in mathematics which is widely applied in network analysis. The growing need for information transfer has made networks and network architecture more crucial. By representing networks as various graph structures, we can utilize different graph theory concepts to examine the interactions and communications within the network. Domination in graphs is a renowned branch of graph theory. Every communication network may be seen as a graph by vertices denoting stations whereas edges the connections among stations. For network designers to rebuild a communication network once certain stations or communication links collapse, network stability is a crucial consideration. The number of non-operational nodes and the largest possible order of the persisting sub network, within which contacts are even now possible, are two crucial variables in the study of a communication network's susceptibility. Communication networks that are more stable or less vulnerable are what we need. For network architects, a communication network's dependability is paramount. The degree of vulnerability of communication networks can be determined using a variety of graph theoretic characteristics. The integrity idea was proposed by C. A. Barefoot et al. [2] and was given as : $I(G) = \min\{|S| + m(G-S): S \subset V(G)\}$, and $m(G-S)$ represents maximum order among the components of $G-S$. Further results on integrity were given in [4,7,8,9]. Integrity is one of the important graph characteristics that gauge a network's susceptibility; it provides a sense of the remaining interconnected subnetworks in the event that the network fails. In communication networks, where the nodes and the connections between any two nodes are represented by the graph's vertices and edges respectively, graph theory is important. Damage resulting from the failure of any network component interrupts the operation of a communication network. A network's vulnerability is defined as its ability to withstand any disturbance or failure. Quantities including the number of non-functioning parts, the size of the largest subnetwork that is still operational, and the number of remaining sub networks are used to quantify a network's susceptibility. One essential component of a network that keeps all of its nodes informed is a dominating set. If such a set is removed from the network, the network will suffer significant harm. One of the vulnerability parameters to measure the performance of communication network is domination integrity. Sundareswaran and Swaminathan [12] proposed the domination integrity of graphs: $DI(G) = \min\{|S| + m(G-S)\}$, in which S represents dominating set of G whereas $m(G-S)$ stands for the maximum order among components of $(G-S)$. Domination integrity of several graphs were found in [11,13]. For a variety of graph types and graph operations, domination integrity has been investigated. Harary and Haynes [10] first put up the idea of double domination. Double domination integrity is the study of the combination

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of two concepts, namely integrity and double domination in graphs. It is a new measure of vulnerability in graphs and was introduced in [6]. Also, it finds an application in PMU placement problem [5]. The double domination integrity of certain graphs is presented in this article. Further, an application of double domination integrity in real world is suggested.

3. DOUBLE DOMINATION INTEGRITY OF A GRAPH

Definition 3.1 [10] A dominating set S of a graph $G = (V, E)$ is a **double dominating set** of G if at least two vertices of S dominate every vertex in V. The **double domination number** of G represented as $\gamma_{dd}(G)$ means the smallest cardinality among the double dominating sets of G.

Definition 3.2 [6] The **double domination integrity** of a connected graph G is represented as $DDI(G)$ which is given by $DDI(G) = \min\{|Y| + m(G-Y): Y$ is a double dominating set of $G\}$ whereas $m(G-Y)$ stands for the maximum order among the components of $G-Y$.

Definition 3.3 [6] A double dominating set Y of G is a **double domination integrity set** or ***DDI*-set** of G if $|Y| + m(G-Y)$ is minimum. Here, Y represents double dominating set of G whereas $m(G-Y)$ indicates maximum order among components of $G-Y$.

4. DOUBLE DOMINATION INTEGRITY OF SOME SPECIAL GRAPHS

Definition 4.1 [3] The graph $W_n = K_1 + C_n$ is known as a **Wheel graph**. The vertex x of degree n is the central vertex and the vertices on Cycle C_n are the rim vertices.

Theorem 4.2 $$DDI(W_n) = \begin{cases} n & if\ n = 3,4 \\ \frac{n}{3} + 3 & if\ n \equiv 0(mod\ 3); n > 3 \\ \frac{n-1}{3} + 4 & if\ n \equiv 1(mod\ 3); n \geq 7 \\ \frac{n-2}{3} + 4 & if\ n \equiv 2(mod\ 3); n \geq 5 \end{cases}$$

Proof: W_n contains $n+1$ vertices. Let x be the central vertex and $u_1, u_2, \dots, u_n$ denote rim vertices of Wheel graph W_n. Let G be the graph W_n.

Case:(i) Since $W_3 \cong K_4$, we get $DDI(W_3) = 4$

Case:(ii) $n = 4$

Clearly, $Y = \{u_1, u_4, u_5\}$ and $Y = \{u_2, u_3, u_5\}$ are the DDI-sets of W_4 since $|Y|$ is same and $m(G-Y) = 1$ in both the situations. Hence, $DDI(W_4) = 4$.

Case:(iii) $n \equiv 0(mod\ 3); n > 3$

Let Y be any double dominating set of G. Since x is incident with all the other vertices of G, $x \in Y$.

Subcase:(a) When $n = 6$, let us choose either $Y_1 = \{u_1, u_3, u_5\} \cup \{x\}$ or $Y_2 = \{u_2, u_5\} \cup \{x\}$ as double dominating sets of G such that $|Y_1| = 4$ and $|Y_2| = 3$. Since $G - Y_1$ contains three isolated vertices, $m(G - Y_1) = 1$ and $G - Y_2$ is a disconnected graph containing two components, each of which is a Path P_2. Thus, $m(G - Y_2) = 2$. Now, $|Y_1| + m(G - Y_1) = 5$ and $|Y_2| + m(G - Y_2) = 5$. Thus, Y_1 and Y_2 are DDI-sets of G. Hence, $DDI(G) = 5 = \frac{n}{3} + 3$ for $n = 6$.

Subcase:(b) When $n > 6$ and $n \equiv 0(mod\ 3)$.

Consider $Y = \{u_{i+1} / i = 3m + 1$ for $m = 0, 1, 2, \dots\}$ such that $|Y| = \frac{n}{3}$. Obviously, $Y' = Y \cup \{x\}$ is a double dominating set of G having $|Y'| = \frac{n}{3} + 1$. Eliminating Y' from G provides components of order two. That is, $m(G - Y') = 2$. Thus, $|Y'| + m(G - Y') = \frac{n}{3} + 3$. No double dominating set Y'' of G exists in which $|Y''| + m(G - Y'') < |Y'| + m(G - Y')$. Hence Y' is the DDI-set of G. Therefore, $DDI(G) = |Y'| + m(G - Y') = \frac{n}{3} + 3$.

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Case:(iv) $n \equiv 1(mod\ 3); n \geq 7$. Let Y be any DDI-set of G.

Subcase:(a) When $n = 7$ and 10, there are two possibilities of choosing the DDI-set. Since x is incident with all the other vertices of G, to obtain $|Y|$ minimum, choose x belongs to Y . In addition to x, Y contains the alternate vertices from the rim vertices or $Y = \{u_{i+1}/i = 3m + 1 \text{ for } m = 0, 1, 2, \dots\} \cup \{u_n\} \cup \{x\}$. In both the circumstances, $|Y| + m(G - Y) = \frac{n-1}{3} + 2 + 2 = \frac{n-1}{3} + 4$.

Subcase:(b) When $n > 10$. Y can be chosen as $\{u_{i+1}/i = 3m + 1 \text{ for } m = 0, 1, 2, \dots\} \cup \{u_n\} \cup \{x\}$ such that , $|Y| = \frac{n-1}{3}$+2 and $m(G - Y) = 2$. Y is the unique DDI-set of G. Hence $DDI(G) = \frac{n-1}{3} + 4$.

Case:(v) $n \equiv 2(mod\ 3); n \geq 5$. Consider Y as any double dominating set of G.

Subcase:(a) When $n = 5$, we can choose Y in two ways. $Y' = \{u_1, u_3, u_5\} \cup \{x\}$ or $Y'' = \{u_2, u_5\} \cup \{x\}$. Now, $|Y'| = 4$, $m(G - Y') = 1$, $|Y''| = 3$, $m(G - Y'') = 2$. Thus, $|Y'| + m(G - Y') = |Y''| + m(G - Y'') = 5$ is minimized. Hence, Y' and Y'' become the DDI-sets of G. Therefore, $DDI(G) = 5$.

Subcase:(b) When $n = 8$, consider the double dominating sets $Y' = \{x\} \cup \{u_2, u_5, u_8\}$ and $Y'' = \{x\} \cup \{u_1, u_3, u_5, u_7\}$ such that $|Y'| = 4$ and $|Y''| = 5$. Now, $m(G - Y') = 2$ whereas $m(G - Y'') = 1$. Y' and Y'' are both DDI-sets of G since $|Y'| + m(G - Y') = |Y''| + m(G - Y'')$ is minimum. Therefore, $DDI(G) = 6$.

Subcase:(c) When $n \equiv 2(mod\ 3); n > 8$. Define $Y = \{x\} \cup \{u_{i+1}/i = 3m + 1 \text{ for } m = 0, 1, 2, \dots\}$ such that $|Y| = \frac{n-2}{3}$+1+1. Clearly, Y is a double dominating set of G. Removing Y from G provides a disconnected graph with components which are either isolated vertex or Path P_2. Thus, $m(G - Y) = 2$. There is no other double dominating set Y^* of G with $|Y^*| + m(G - Y^*) < |Y| + m(G - Y)$. Hence Y becomes the DDI-set of G. Thus $DDI(G) = \frac{n-2}{3} + 3$.

Definition 4.3 [1] For any positive integer $n \geq 2$, an **Octopus graph** is constructed by attaching a Fan graph F_n to a Star graph $K_{1,n}$ by sharing a common vertex.

Theorem 4.4 For $n \geq 3$,

$$DDI(O_n) = \begin{cases} 2n & if\ n = 3, 4 \\ 2n - 1 & if\ n = 5 \\ \dfrac{4n}{3} + 3 & if\ n \equiv 0(mod\ 3); n \neq 3 \\ \dfrac{4n-1}{3} + 4 & if\ n \equiv 1(mod\ 3); n \neq 4 \\ \dfrac{4n-2}{3} + 4 & if\ n \equiv 2(mod\ 3); n \neq 5 \end{cases}$$

Proof: Let the end vertices of O_n be $v_1, v_2, \dots, v_n$ and let u represents the support vertex of O_n. Let $Y = \{u, v_1, v_2, \dots, v_n\}$. Let the Path P_n in O_n is represented as $u_1, u_2, u_3, \dots, u_n$. For choosing the DDI-set of O_n, the following cases arise.

Case:(i) $n = 3$

Obviously, $Y_1 = Y \cup \{u_2\}$ forms the double dominating set of O_n and $m(O_n - Y_1) = 1$. Therefore, Y_1 is the DDI-set of O_n. Hence, $DDI(O_n) = |Y_1| + m(O_n - Y_1) = 3 + 1 + 1 + 1 = 6 = 2n$.

Case:(ii) $n = 4$

Let the vertices of Path P_4 in O_4 be u_1, u_2, u_3, u_4. Let $Y_1 = Y \cup \{u_1, u_3\}$ and $Y_2 = Y \cup \{u_2, u_4\}$. Then $|Y_1| = 7$ and $|Y_2| = 7$. Here both Y_1 and Y_2 are double dominating sets of O_n. Removal of Y_1 or Y_2 from O_n provides two isolated vertices. Therefore, $m(O_n - Y_1) = 1$ and $m(O_n - Y_2) = 1$. Hence, $DDI(O_n) = |Y_1| + m(O_n - Y_1) = |Y_2| + m(O_n - Y_2) = 8$.

Case:(iii) $n = 5$

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Clearly, $Y_1 = Y \cup \{u_2, u_4\}$ is the unique DDI-set for O_n since $|Y_1| + m(O_n - Y_1)$ is minimum. $|Y_1| = 6 + 2$ and $m(O_n - Y_1) = 1$. Therefore, $DDI(O_n) = 9$.

Case:(iv) $n = 7$

It is obvious that $Y' = \{u, v_1, v_2, v_3, v_4, v_5, v_6, v_7, u_2, u_4, u_6\}$ is the DDI-set of O_n because no other double dominating set Y'' of O_n exists with $|Y''| + m(O_n - Y'') < |Y'| + m(O_n - Y')$. Thus $|Y'| = 11$ and $m(O_n - Y') = 1$. Hence $DDI(O_n) = |Y'| + m(O_n - Y') = 12 = \frac{4n}{3} + 3$.

Case:(v) $n \equiv 0(mod\ 3); n \neq 3$

Y is a subset of the DDI-set of O_n such that $|Y| \leq \frac{4n}{3} + 1$. Define the set $Y' = \{u_{i+1}/i = 3m + 1 \text{ for } m = 0, 1, 2, \ldots\}$ where u_{i+1} represents the vertices selected at intervals determined by the formula $i = 3m + 1$. Let $Y_1 = Y \cup Y'$. Clearly, Y_1 forms a double dominating set of O_n. Removing Y_1 from O_n produces a graph with two components of order 1 and the remaining components of order 2. So, $m(O_n - Y_1) = 2$. The size of Y_1 is $|Y_1| = |Y| + |Y'| = \frac{4n}{3} + 1$. Since $|Y_1| + m(O_n - Y_1)$ is minimized, Y_1 is the DDI-set of O_n. Therefore, $DDI(O_n) = |Y_1| + m(O_n - Y_1) = \frac{4n}{3} + 3$

Case:(vi) $n \equiv 1(mod\ 3); n \neq 4, 7$

Describe the set $Y' = \{u_n, u_{i+1}/i = 3m + 1\}$ for $m = 0, 1, 2, \ldots$ such that $|Y'| = \frac{n-1}{3} + 1$. Y is contained in the DDI-set of O_n. Let $Y_1 = Y \cup Y'$. $|Y_1| = n + 1 + \frac{n-1}{3} + 1 = \frac{4n-1}{3} + 2$. Y_1 is a double dominating set of O_n as Y_1 dominates all the vertices of O_n at least twice. Eliminating Y_1 from $V(O_n)$ provides a graph containing components with order either 1 or 2. So, $m(O_n - Y_1) = 2$. Clearly, $|Y_1| + m(O_n - Y_1)$ is minimum. Hence, Y_1 forms the DDI-set of O_n. Therefore, $DDI(O_n) = |Y_1| + m(O_n - Y_1) = \frac{4n-1}{3} + 2 + 2 = \frac{4n-1}{3} + 4$.

Case:(vii) $n \equiv 2(mod\ 3); n \neq 5$

Let $Y' = \{u_n, u_{i+1}/i = 3m + 1 \text{ for } m = 0, 1, 2, \ldots\}$ and $Y'' = \{u_{n-1}, u_{i+1}/i = 3m + 1 \text{ for } m = 0, 1, 2, \ldots\}$. Y is a subset of the DDI-set of O_n. Let $Y_1 = Y' \cup Y$ and $Y_2 = Y'' \cup Y$. So, $|Y_1| = n + 1 + \frac{n-2}{3} + 1 = \frac{4n-2}{3} + 2$ and $|Y_2| = \frac{4n-2}{3} + 2$. Y_1 and Y_2 are double dominating sets of O_n for which $|Y_1| + m(O_n - Y_1)$ and $|Y_2| + m(O_n - Y_2)$ are minimum. $m(O_n - Y_1) = m(O_n - Y_2) = 2$. Y_1 and Y_2 are both DDI-sets of O_n. Therefore, $DDI(O_n) = |Y_1| + m(O_n - Y_1) = |Y_2| + m(O_n - Y_2) = \frac{4n-2}{3} + 2 + 2 = \frac{4n-2}{3} + 4$.

Definition 4.5 [15] A **Fan graph** is formed by connecting a central vertex x to all the vertices of Path P_n.

Theorem 4.6 The double domination integrity of Fan graph F_n is

$$DDI(F_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 2 & for\ 2 \leq n \leq 7 \\ \left\lceil \frac{n}{3} \right\rceil + 3 & for\ n > 7 \end{cases}$$

Proof: Let u be the central vertex and $u_1, u_2, \ldots, u_n$ be vertices of Path P_n in the Fan graph F_n. Let Y be a double dominating set of F_n. Let G be the graph F_n.

Case:(i) $2 \leq n \leq 7$

The double dominating set Y and the values of $m(G - Y)$ for $n = 2$ to 7 and the value of $|Y| + m(G - Y)$ for which $|Y| + m(G - Y)$ is minimum is given in the following table.

n	Y	$\lvert Y\rvert$	$m(G - Y)$	$\lvert Y\rvert + m(G - Y)$
2	$\{u_1, u_2\}$ or $\{u_2, u\}\{u, u_1\}$	2	1	3
3	$\{u, u_2\}$	2	1	3

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4	$\{u, u_2, u_4\}$ or $\{u, u_1, u_3\}$ or $\{u, u_2, u_3\}$	3	1	4
5	$\{u, u_2, u_4\}$	3	1	4
6	$\{u, u_2, u_4, u_6\}$ or $\{u, u_1, u_3, u_5\}$	4	1	5
6	$\{u, u_2, u_5\}$	3	2	5
7	$\{u, u_2, u_4, u_6\}$	4	1	5

The set Y forms the DDI-set of G. Hence for $n = 2$ to 7,

$$DDI(G) = \begin{cases} 3 & if\ n = 2,3 \\ 4 & if\ n = 4,5 \\ 5 & if\ n = 6,7 \end{cases}$$

$$= \left\lfloor \frac{n}{2} \right\rfloor + 2 \quad for\ \ 2 \leq n \leq 7$$

Case:(ii) $n > 7$

Subcase: (a) For $n = 8$ to $11, 13$, the DDI-set can be chosen in the following ways. $Y_1 = \{u\} \cup \{u_{i+1}/i = 3m + 1$ for $m = 0, 1, 2, \dots\}$ and $Y_2 = \{u\} \cup \{u_{2m}/1 \leq m \leq 5\}$ are described in the following tables.

n	Y_1	$\lvert Y_1 \rvert$	$m(G - Y_1)$	$\lvert Y_1 \rvert + m(G - Y_1)$
8	$\{u, u_2, u_5, u_7\}$	4	2	6
9	$\{u, u_2, u_5, u_8\}$	4	2	6
10	$\{u, u_2, u_5, u_8, u_{10}\}$	5	2	7
11	$\{u, u_2, u_5, u_8, u_{10}\}$	5	2	7
13	$\{u, u_2, u_5, u_8, u_{11}, u_{13}\}$	6	2	8

n	Y_2	$\lvert Y_2 \rvert$	$m(G - Y_2)$	$\lvert Y_2 \rvert + m(G - Y_2)$
8	$\{u, u_2, u_4, u_6, u_8\}$	5	1	6
9	$\{u, u_2, u_4, u_6, u_8\}$	5	1	6
10	$\{u, u_2, u_4, u_6, u_8, u_{10}\}$	6	1	7
11	$\{u, u_2, u_4, u_6, u_8, u_{10}\}$	6	1	7
13	$\{u, u_2, u_4, u_6, u_8, u_{10}, u_{12}\}$	7	1	8

The above defined sets Y_1 and Y_2 are DDI-sets of G. Therefore,

$$DDI(G) = \begin{cases} 6 & for\ n = 8,9 \\ 7 & for\ \ n = 10,11 \\ 8 & for\ n = 13 \end{cases}$$

Subcase: (b) $n > 11, n \neq 13$

Let Y be a double dominating set of G. Since u is incident with all the other vertices of G, choose u belongs to Y. Consider $Y = \{u\} \cup \{u_{i+1}/i = 3m + 1$ for $m = 0, 1, 2, \dots\}$ such that $|Y| = \left\lceil \frac{n}{3} \right\rceil + 1$ whereas $m(G - Y) = 2$. Thus, $|Y| + m(G - Y) = \left\lceil \frac{n}{3} \right\rceil + 3 \geq DDI(G)$. We claim that $DDI(G) = |Y| + m(G - Y)$. Describe $Y_1 = \{u\} \cup \{u_2, u_4, \dots, u_{n-1}\}$ if n is odd and $Y_2 = \{u\} \cup \{u_2, u_4, \dots, u_n\}$ if n is even. If we consider the double dominating sets Y_1 and Y_2, we get $m(G - Y_1) =$

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$m(G-Y_2)=1$ (which is minimum). But $|Y_1|+m(G-Y_1)>|Y|+m(G-Y)$ and $|Y_2|+m(G-Y_2)>|Y|+m(G-Y)$. Hence $|Y|+m(G-Y)$ is minimized. Thus, Y is the DDI-set of G. Hence $DDI(G)=|Y|+m(G-Y)=\left\lceil\frac{n}{3}\right\rceil+3$

Definition 4.7 [14] A graph obtained by series of interconnected m copies of n stars by linking one leaf from each is called a **Firecracker graph**.

Theorem 4.8 For $m\geq 2$, $DDI\big(F(m,n)\big)=mn$ if $n=2$

Proof: $F(m,n)$ contains mn vertices. Since the graph $F(m,n)$ contains only end and support vertices for $n=2$, $DDI\big(F(m,n)\big)=mn$

Theorem 4.9 For $2\leq m\leq 5, n\geq 3$,

$$DDI\big(F(m,n)\big)=\begin{cases}2n & if\ m=2\\ mn-1 & if\ m=3,4\\ mn-2 & if\ m=5\end{cases}$$

Proof: Let $u_{ij}(1\leq i\leq m, 1\leq j\leq n)$ represent vertices of m copies of n star, such that $u_{11},u_{21},\dots,u_{m1}$ be the vertices of Path P_m in $F(m,n)$. Let G denote the graph $F(m,n)$. Let $Y=\{u_{ij}/1\leq i\leq m, 2\leq j\leq n\}$ be the set containing end and support vertices of G. Thus $|Y|=m(n-1)$.

Case:(i) $m=2$

Y belongs to the DDI-set of G. $Y'=Y\cup\{u_{11}\}$ as well as $Y''=Y\cup\{u_{21}\}$ are double dominating sets of G having minimum cardinality. $|Y'|=2(n-1)+1$, $|Y''|=2(n-1)+1$ and $m(G-Y')=m(G-Y'')=1$. Hence Y' and Y'' are DDI-sets of G. Therefore, $DDI(G)=|Y'|+m(G-Y')=|Y''|+m(G-Y'')=2(n-1)+1+1=2n$

Case:(ii) $m=3$

Obviously, $Y'=Y\cup\{u_{21}\}$ is the only double dominating set of G having minimum order. $|Y'|=3(n-1)+1$. Removal of Y' from G results in a disconnected graph with two isolated vertices. Thus, $m(G-Y')=1$; which is minimum. Hence, Y' is the DDI-set of G and so $|Y'|+m(G-Y')=3(n-1)+1+1$. Thus, $DDI(G)=3(n-1)+2=3n-1=mn-1$.

Case:(iii) $m=4$

Let $Y'=Y\cup\{u_{11},u_{31}\}$ and $Y''=Y\cup\{u_{21},u_{41}\}$. Then $|Y'|=4(n-1)+2$ and $|Y''|=4(n-1)+2$. Clearly, Y' and Y'' are double dominating sets of G. Eliminating the vertices of Y' or Y'' from G gives a disconnected graph containing two isolated vertices. Therefore, $m(G-Y')=1$ and $m(G-Y'')=1$. Hence $|Y'|+m(G-Y')=|Y''|+m(G-Y'')$ is minimized. Thus, both Y' and Y'' are DDI-sets of G. Therefore, $DDI(G)=4(n-1)+2+1=4n-1=mn-1$.

Case:(iv) $m=5$

Let $Y'=Y\cup\{u_{21},u_{41}\}$. Thus $|Y'|=5(n-1)+2=5n-3$. Y' is a double dominating set of G. Also, $m(G-Y')=1$. Thus $|Y'|+m(G-Y')=5n-2$. There exists no other double dominating set Y'' of G such that $|Y''|+m(G-Y'')<|Y'|+m(G-Y')$. Thus Y' is the DDI-set of G. Therefore, $DDI(G)=|Y'|+m(G-Y')=5n-2=mn-2$

Theorem 4.10 For $m>5$ and $n\geq 3$,

$$DDI(F(m,n))=\begin{cases}mn-2\left(\frac{m-3}{3}\right) & if\ m\equiv 0(mod\ 3)\\ mn-2\left(\frac{m-4}{3}\right) & if\ m\equiv 1(mod\ 3)\\ mn-2\left(\frac{m-5}{3}\right)+1 & if\ m\equiv 2(mod\ 3)\end{cases}$$

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Proof: Let $u_{ij}(1 \leq i \leq m, 1 \leq j \leq n)$ be the vertices of m copies of n star such that $u_{11}, u_{21}, \ldots, u_{m1}$ denote the vertices of Path P_m in $F(m,n)$. Let G represents the graph $F(m,n)$. Let $Y = \{u_{ij}/1 \leq i \leq m, 2 \leq j \leq n\}$ be the set containing end and support vertices of G. Then $|Y| = m(n-1)$. Let $u_{11} = u_1, u_{21} = u_2, \ldots, u_{m1} = u_m$.

Case:(i) $m \equiv 0(mod\ 3)$

Define u_{i+1} to represent the vertices selected at intervals determined by the formula $i = 3m+1$ for $m = 0,1,2, \ldots$. Let $Y' = \{u_{i+1}/i = 3m+1;\ m = 0,1,2, \ldots\}$. $Y_1 = Y \cup Y'$ forms the double dominating set of G having minimum cardinality $|Y_1| = \frac{m}{3}$. In such case, $m(G - Y_1) = 2$. If we take into account other double dominating set Y_2 of G having $m(G - Y_2) = 1$, we get $|Y_2| > |Y_1|$ for $n > 6$. Hence Y_1 becomes the DDI-set of G for $n > 6$. For $n = 6$, when $m(G - Y_2) = 1$ with $Y_2 = Y \cup \{u_1, u_3, u_5\}$ we get $|Y_2| = m(n-1) + 3$. Thus, for $n = 6$, both Y_1 and Y_2 are DDI-sets of G; since $|Y_1| + m(G - Y_1) = |Y_2| + m(G - Y_2)$. Therefore, $DDI(G) = m(n-1) + \frac{m}{3} + 2 = mn - \frac{2m}{3} + 2 = mn - 2\left(\frac{m-3}{3}\right)$

Case:(ii) $m \equiv 1(mod\ 3)$

Define $Y' = \{u_n, u_{i+1}/i = 3m+1;\ m = 0,1,2, \ldots\}$ with $|Y'| = \frac{m+2}{3}$. Clearly, $Y_1 = Y \cup Y'$ is the double dominating set of G having minimum cardinality. Then $|Y_1| = m(n-1) + \frac{m+2}{3}$. In this case, $m(G - Y_1) = 2$. Since there exists no other double dominating set Y_2 with $|Y_2| + m(G - Y_2) < |Y_1| + m(G - Y_1)$, Y_1 is the DDI-set of G. Therefore, $DDI(G) = m(n-1) + \frac{m+2}{3} + 2 = mn - \left(\frac{3m-m-2-6}{3}\right) = mn - 2\left(\frac{m-4}{3}\right)$

Case:(iii) $m \equiv 2(mod\ 3)$

Consider $Y' = \{u_{i+1}/i = 3m+1$ for $m = 0,1,2, \ldots\}$ and $Y'' = \{u_{i+1}, u_{n-1}/i = 3m+1$ for $m = 0,1,2, \ldots$ and $i+1 < n\}$. Thus $|Y'| = |Y''| = \frac{m+1}{3}$. Let $Y_1 = Y \cup Y'$ and $Y_2 = Y \cup Y''$. So, $|Y_1| = |Y_2| = m(n-1) + \frac{m+1}{3}$. Removing the vertices of Y_1 or Y_2 from G results with $m(G - Y_1) = m(G - Y_2) = 2$. If we take into account other double dominating set Y_3 of G, we get $|Y_3| + m(G - Y_3) > |Y_1| + m(G - Y_1)$ and $|Y_3| + m(G - Y_3) > |Y_2| + m(G - Y_2)$. Thus, $|Y_1| + m(G - Y_1) = |Y_2| + m(G - Y_2)$ is minimized. Hence, Y_1 and Y_2 are DDI-sets of G. Thus, $DDI(G) = m(n-1) + \frac{m+1}{3} + 2 = mn - \left(\frac{3m-m-1-6}{3}\right) = mn - 2\left(\frac{m-5}{3}\right) + 1$.

5. APPLICATION OF DOUBLE DOMINATION INTEGRITY IN POWER GRIDS

Double domination integrity is essential to resilient systems, especially power grids, in order to preserve operation and reduce disturbance in the case of failures. In power grids, double domination integrity helps preserve power even in the event of a single source failure by guaranteeing that many power sources can provide electricity to vital nodes (such as hospitals or emergency services) since power is supplied to a particular node by at least two sources. Integrity concept is useful in the case that power can be supplied continuously even in the event of a transmission line or substation failure by providing backup routes. Public safety and infrastructure trust can be greatly enhanced by making sure that vital services continue to run even in the face of interruptions. In general, double domination integrity guarantees that vital systems are robust and able to continue operating even in the event of breakdowns, which is crucial for both public safety and the efficient operation of contemporary infrastructure.

6. CONCLUSION

This work presents the computation of double domination integrity of some special graphs. Also, an application of double domination integrity in real world is given.

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