Journal of Information Systems Engineering and Management
2025, 10(11s)
e-ISSN: 2468-4376
https://www.jisem-journal.com/

Research Article

Robust Control and Synchronization of Fractional-Order Unified Chaotic Systems

Nawaf Alsowait[1], Thwiba A. Khalid[2,3*], Fatima Alnoor[4], Marwa Mutasim Mohammed[4], Nidal.E.Taha[5]
[1]*Department of Mathematics, College of Science, Northern Border University, Arar, Saudi Arabia.*
[2] *Department of mathematics, Faculty of Science, Al-Baha University, Al-Baha 65526, Saudi Arabia, Saudi Arabia*
[3]*Academy of Engineering and Medical Sciences, Department of Mathematics, Khartoum, Sudan*
[4]*Department of Mathematics, College of Science, Sudan University of Science and Technology, khartoum, Sudan.*
[5]*Department of Mathematics, College of Science, Qassim University, Buraidah 51452, Saudi Arabia*
* *Corresponding author: tabdulrhman@bu.edu.sa*

ARTICLE INFO

Received: 14 Nov 2024

Revised: 26 Dec 2024

Accepted: 14 Jan 2025

ABSTRACT

This work examines the understanding and regulation of chaotic systems via fractal system dynamics. The criteria for the asymptotic stability of the equilibrium point are examined. This signifies that the point is stable and demonstrates local attraction, which is essential for the system's behavior over time. The independent uncertainty in the system, denoted by αj, may vary over time but is constrained by a constant δ > 0. This factor is crucial for comprehending the system's responsiveness to varying environments. The system's stability is further analyzed through the negative semi-definiteness of the fractional derivative of the Lyapunov function, indicating that the system's origin stays stable given specific beginning circumstances. The study presents a thorough framework for analyzing and controlling chaotic systems using fractal system dynamics, emphasizing stability and the impact of uncertainty.

Keywords: Fractional-Order Systems, Chaotic Synchronization, Control Systems, Stability Analysis, Error Dynamics.

INTRODUCTION

The research emphasizes the importance of chaotic systems across many fields, highlighting their complex behavior and sensitivity to initial conditions. This complexity requires advanced analytical and control methodologies. Fractional system dynamics is presented as an effective tool for modeling and regulating chaotic systems. These dynamics provide a more accurate depiction of real-world events than traditional integer order models [1][2]. The introduction discusses the importance of stability analysis in chaotic systems. It shows how derivatives of the fractional system can evaluate the stability of equilibrium points, which is essential for ensuring the reliability of the system dynamics [3][4]. The authors intend to create a model for analyzing and regulating the stability of chaotic systems using fractional system dynamics. This entails identifying the basic requirements for asymptotic stability and examining the effect of uncertainty on the system dynamics. Fractional calculus has emerged as an important mathematical tool for modeling many real-world phenomena in recent years [5][6][7]. It has been increasingly considered important by many scientists in various fields of engineering and science, including control engineering, electrochemistry, electromagnetism, biological sciences, and diffusion processes. Fractal order dynamics is proposed as a useful method for modeling and controlling chaotic systems [8][9]. This creates a foundation for further study of chaotic systems that can be described by fractal order equations. In the first part, important ideas such as the Riemann–Liouville derivative and other stability theorems are explained. These theorems tell us how to judge local and global stability in fractal order systems [10][11]. The theorems state the basic requirements for systems to exhibit asymptotic local and global stability, highlighting the importance of Lyapunov functions in stability assessments. The final section of the study examines the concept of synchronization in chaotic systems characterized by fractal order dynamics. The authors create a controller that ensures that the error between the master and slave systems approaches zero and establish a framework for evaluating the stability and management of chaotic systems using fractal order dynamics. This entails defining asymptotic stability criteria and examining the effect of uncertainty on system dynamics. The study describes a method for using finite matrix properties and Lyapunov stability criteria to assess the long-term stability of linked systems. The traditional Riemann–Liouville definition of the fractional derivative has problems with the nonzero derivative of the constant. To fix these problems, Jomari came up with a different approach called the local modified Riemann–Liouville definition. Section 3 will analyze the results of Streep. We use the potential Lyapunov function to demonstrate controlled chaos for two fruit points using fractional control graphs (FCGs) of fractional order [12]. This shows that the original system is stable under certain conditions, in particular when α = 1. Using the unit design, skillful control may help in the efficient synchronization of chaotic systems of fractional order, which will greatly improve communications through noise. The results enhance chaos theory and its various applications by providing methods relevant to non-classical electronics [20][21].

PRELIMINARIES

This research will concentrate on a dynamical system that fulfils We will assume that the function $(\phi(u,s))$ satisfies the stipulated criteria.

$$u^{\cdot} = \phi(u,s), u(s_0) = u_0 \ , \ u \in \mathbb{R}^n \tag{1}$$

Definition 1. [18] The equilibrium points $u^* = 0$ of (1)is stable (Pertaining to the Lyapunov notion) at $s = s_0$ if for any $\varsigma > 0$ there exists a $\varrho(s_0) > 0$ such that

$$\|u(s_0)\| < \varsigma \Rightarrow \| u(s) \| < \varrho \qquad \forall st \geq s_0 \tag{2}$$

Definition 1 is independent of t_0, hence equation (2) is applicable for all values of s_0. The definition of asymptotic stability is articulated as stated below:

Definition 2. [19] An equilibrium points $u^* = 0$ of (1) is asymptotically stable at $s = s_0$ if

1. $u^* = 0$ is stable, and
2. $u^* = 0$demonstrates regional attraction; specifically, it exists $\varsigma(s_0)$ in such a manner that

$$\| u(t_0) \| < \delta \Rightarrow \lim_{s \to \infty} u(t) = 0 \tag{3}$$

The previously described definition indicates that asymptotic stability at time 0 is invariant. Uniform asymptotic stability requires:

1. $u^* = 0$ to be uniformly stable, and
2. $u^* = 0$ The model exhibits uniform local attractiveness, signifying the existence of $a\ \varrho$ that is independent on s_0, for which equation (3) holds true. Moreover, uniform convergence in equation(3)is essential [17].

Let $n \in \mathbb{Z}^+$and D_t^n t denote the classical derivative of order n.

Comparative strengthIn the next portions of the work, the symbol D_t^m t is utilized to denote the Riemann-Liouville derivative of order m [30].

Definition 3. [16] The fractional-order integral and derivative operators can be defined as

$$D_s^{\alpha_1} = \begin{cases} \frac{d^{\alpha_1}}{ds^{\alpha_1}} & ,\alpha_1 > 0 \\ 1 & ,\alpha_1 = 0 \\ \int_a^s (ds)^{-\alpha_1} & ,\ \alpha_1 < 0 \end{cases} \tag{4}$$

Where α_1 is the fractional order of the operator, represented as a complex number.

Definition4. [15] The α_1^{th} The Riemann-Liouville fractional-order integral of the function $f(s)$ is defined as

$$_{s_0}I_s^{\alpha_1} f(s) = \frac{1}{\Gamma(\alpha-1)} \int_{s_0}^{S} \frac{f(\omega)}{(t-\omega)^{(1-\alpha_1)}} d\omega \tag{5}$$

where s_0 is the initial time and $\Gamma(\alpha_1)$ is Gamma function which is determined as

$$\Gamma(\alpha_1) = \int_0^\infty e^{-s} e^{\alpha_1 - 1} ds \tag{6}$$

Where α_1 is the operator of the Gamma function.

Definition 5. [14] Let $n-1<\alpha_1\leq n, n\in N$, the Riemann–Liouville fractional derivative of order $\alpha_1$of $f(t)$ is described by

$$_{s_0}D_s^{\alpha_1}f(s)=\frac{d^{\alpha_1}f(s)}{ds^{\alpha_1}}=\frac{1}{\Gamma(n-\alpha_1)}\frac{d^n}{ds^n}\int_{s_0}^{S}\frac{f(\omega)}{(t-\omega)^{(1-\alpha_1)}}d\omega \quad (7)$$

Remark 1. [32] In the Riemann-Liouville fractional derivative (Eq. (7)), integration precedes differentiation; hence, the fractional derivative of a constant is not zero in this framework.

Definition6. [13]. The Caputo fractional derivative of order$\alpha_1$of a continuous functional $f(s)$ is described by

$$_{s_0}D_s^{\alpha_1}f(s)=\begin{cases}\frac{1}{\Gamma(m-\alpha_1)}\int_{s_0}^{S}\frac{f^{(m)}(\omega)}{(s-\omega)^{\alpha_1-m+1}}d\omega, & m-1<\alpha_1<m\\ \frac{d^m f(s)}{ds^m}, & \alpha_1=m\end{cases} \quad (8)$$

Where m is the first integer number, bigger than α_1 .

Lemma2.[22] Consider $x=0$ as equilibrium of fractional-order

non- autonomous system

$$D^x=f(x,s) \quad (9)$$

Where $f(x,s)$ fulfils the Lipschitzian condition with coefficient $l¿0$ *and* $\alpha_1\in(0,1)$.Suppose that there exists the Lyapunov functional $V(s,x(s))$ guaranteeing

$$\beta_1\parallel x\parallel^{\beta}\leq V(x,s)\leq\beta_2\parallel x\parallel \quad (10)$$

$$V(s,x)\leq-\beta_3\parallel x\parallel \quad (11)$$

Let β_1,β_2,β_3 and β indicate positive benefits, and let $\|.\|$. represent an arbitrary norm. Thus, the equilibrium of dynamics (9) is asymptotically stable. This lemma is pertinent to the Caputo and Riemann-Liouville definitionsMethods

THE PROPOSED FRAMEWORK FOR FRACTIONAL-ORDER CHATICS

This section analyses the synchronisation of two fractional-order chaotic systems, presenting a new integer-order hyperchaotic system and outlining its equations.

$$D^{\mu}v=M(v)v \quad (12)$$

Where $v=(v_1,v_2,\dots,v_n)^s\in\mathbb{R}^n$ Let (v) Let the state vector be denoted as X, $M(v)$

represent the parametric coefficient matrix of states, and 0 signify the fractional commensurate order [24]. The establishment of slavery is

$$D\,\tilde{v}=A(\tilde{v})(\tilde{v})+u(s) \quad (13)$$

Let $u(s)$ denote the controller to be constructed. The equation $e(s)=\hat{x}(s)-u(s)$ is defined. It is feasible to get the error system:

$$D^q e=F(e,u)+u(e,u) \quad (14)$$

Let $F(e,u)$ Let b be defined as the difference between $A(u)$and the square of $\hat{u}$. The synchronisation issue may be transformed into the design of a controller u

$$\lim_{s\to\infty}\|e(s)\|=0$$

To facilitate further analysis, a valuable theorem is introduced [16]. Examine a category of cascade-connected systems characterized by

$$D_{u_1}^{v}=f(u_1) \quad (15)$$

$D^u(u_1,u_2)=A(u_1,u_2)u_2+B(u_1,u_2)\,g(x_1)$ where $u_1\in\mathbb{R}^n$ $u_2\in\mathbb{R}^m, f(0)=0$ *and* $g(0)=0$. $f(u_1)$ and $g(u_1)$ are both C^1 vector fields. $A(u_1,u_2)$ and $B(u_1,u_2)$ are C^m and C^n The coefficient matrix, correspondingly.

Theorem 8. [27] The subsystem $D^v\ u_1 = f(u_1)$ exhibits global asymptotic stability at $u_1 = 0, (2)\ B(u_1, u_2)$ is a bounded matrix and $\lim_{s\to\infty} g(u_1) = 0$ (3) $A(u_1, x_2)$ is a matrix with the following structure:) $A(u_1, u_2)$

$$\begin{pmatrix} A_{11}(.) & A_{12}(.) & \cdots & A_{1n}(.) \\ A_{21}(.) & A_{22}(.) & & \\ \vdots & & \ddots & \vdots \\ \vdots & \vdots & & \\ A_{n1}(.) & \cdots & \cdots\ A_{nn-1}(.) & A_{nn}(.) \end{pmatrix}$$

Where $A_{ij}(.) \leq 0, i = j$, $A_{ij}(.) = -A_{ij}(.)\ i \neq j$ then system (12) is globally asymptotically stable at the equilibrium

$$(U_1, U_2) = (0,0)$$

Proof. Given the subsystem

$$D^V_{U_1} = f(U_1) \tag{16}$$

Exhibits global asymptotic stability alone at -1=0

Require introspection of the subsystem $D^q(u_1, U_2) = A(u_1, u_2)\ u_2 + B(u_1, u_2)g(u_1)$

Given assumption (2), it is evident that $\lim_{s\to\infty} g(u_1) = 0$ Let us investigate the existence of a bounded matrix referred to as $B(x_1, x_2)$ According to Lemma 3, it is clear that $\lim_{s\to\infty}[\,B(u_1, u_2)g(u_1)] = 0$ This indicates that the second element or component in the system

$$D^v(u_1, u_2) = A(u_1, u_2)\ u_2 + B(u_1, u_2)g(u_1) \tag{17}$$

Can be ignored as t approaches zero. To investigate the asymptotic stability of system (12), it is adequate to concentrate on the subsequent model

$$D^q(u_1, u_2) = A(u_1, u_2)\ u_2 \tag{18}$$

We demonstrate that system (18) is globally asymptotically stable at equilibrium using three distinct methodologies cite29. $(u_1, u_2) = (0,0)$

Initially, let us analyse the following equation:

$$A(u_1, u_2)\varphi = \lambda\varphi \tag{19}$$

where λ is one of the eigenvalues of $A(u_1, u_2)$ and φ is a non-zero eigenvector of the matrix (v). Calculate the conjugate transpose and position it on either side of the equation. (19) and we can obtain

$$(A(u_1, u_2)\varphi)^T = \varphi^{-Q} \tag{20}$$

Combine (19) with (20) Furthermore, we may obtain

$$\xi^Q = (PA(u_1, u_2) + A(u_1, u_2)^Q P)\varphi = (\lambda + \lambda^-)\ \varphi^Q P\xi \tag{21}$$

Furthermore, the structure of $A(u_1, u_2)$

$$\begin{pmatrix} A_{11}(.) & A_{12}(.) & \cdots & A_{1n}(.) \\ A_{21}(.) & A_{22}(.) & & \\ \vdots & & \ddots & \vdots \\ \vdots & \vdots & & \\ A_{n1}(.) & \cdots & \cdots\ A_{nn-1}(.) & A_{nn}(.) \end{pmatrix}$$

$$\begin{pmatrix} A_{11}(.) & A_{12}(.) & \cdots & A_{1n}(.) \\ A_{21}(.) & A_{22}(.) & & \\ \vdots & & \ddots & \vdots \\ \vdots & \vdots & & \\ A_{-n1}(.) & \cdots & \cdots\ A_{-nn-1}(.) & A_{nn}(.) \end{pmatrix}$$

The matrix $A(u_1, u_2)$ clearly satisfies the continuity requirement. Lyapunov equation

$$[A(u_1,u_2)P + PA(u_1,u_2)^Q = -Q] \quad (22)$$

where $P = 1$ is the real symmetric identity matrix and

$$\begin{bmatrix} -2\Lambda_{11} & & \\ & -2\Lambda_{22} & \\ & & -2\Lambda_{33} \end{bmatrix} \quad (23)$$

Forms a Hermitian matrix. Furthermore, we can exhibit

$$\Lambda(u_1,u_2)P + P\Lambda(u_1,u_2)^Q = (A(u_1,u_2)P + P\Lambda(u_1,u_2)^Q)^Q \quad (24)$$

Specifically, $\Lambda(u_1,u_2)P + P\Lambda(u_1,u_2)^Q$ is also a Hermitian matrix. Finally, based on the characteristics of positive definite and negative semidefinite matrices, we may derive two inequalities.

$$(\varphi^Q P\Lambda(u_1,u_2) + (\Lambda(u_1,u_2)^H P)\varphi = \varphi^Q(-g)\varphi \leq 0) \quad (25)$$

$$\varphi^Q P\varphi > 0$$

Thus, by combining equation (18), we may obtain

$$(\Theta+\Theta^-)=\frac{\varphi^H(-g)\varphi}{\varphi^Q P\varphi} \leq 0 \quad (26)$$

Clearly, any eigenvalue a of the coefficient matrix $\Lambda(u_1,u_2)$ satisfies the following inequalcity

$$|arg\ (\Theta)| \geq \frac{\pi}{2} > \frac{\beta\pi}{2} \qquad (\Theta< 1) \quad (27)$$

From 7, we may conclude that system (18) is globally asymptotically stable at the equiLibrium point when u1 and u2 are both (0,0). This completes the evidence

A new integer-order hyperchaotic system was introduced. The system's equations are specified as

$$\frac{dx}{dt} = a_1u - a_1y + lx - xu$$

$$\frac{dy}{dt} = b_1x + u - x_2 \quad (29)$$

$$\frac{dz}{dt} = x^2 - c_1z$$

$$\frac{du}{dt} = d_1u$$

where the constants a1, b1, c1, d1 ∈ R. The fractional form of system (29) is described as

$$D^{''}x = a_1u - a_1y + lx - xu$$

$$D^{''}y = b_1x + u - x_2 \quad (30)$$

$$D^{''}z = x^2 - c_1z$$

$$D^{''}u = d_1u$$

where 0 < a_1,≤ 2 The system (30)Exhibits three equilibria.$T_i(x,y,z,u) = 0,1,2$ where $T_0(0,0,0,0)$, $S_1(\psi, \psi^{h/a_1}, b_1, 0)$ and $T_2 = (-(\psi_1, (\psi_1{}^{h/a}, \psi = \sqrt{bc}, a \neq 0, bc > 0$. Furthermore, system (30) possesses the unique equilibrium $T_0(0,0,0,0)$ whena_1, = 0 orb_1c_1,, < 0. Recently, numerous numerical techniques for solving FDS have emerged, including the variational iteration method (VIM), the transfer function approximation method in the frequency domain , Adomian's decomposition method (ADM) and the predictor-corrector method (PECE), which demonstrates enhanced efficiency and is extensively utilized in practical applications [29]. Consequently, the fractional systems in this study are integrated utilizing the PECE framework to handle FDS, namely, Predict, Evaluate, Correct, and Evaluate. The PECE algorithm requires a total of $n = 5 \times 10^4$points and a discretization step of 2×10^{-3}

THE DYNAMIC ANALYSIS

This section addresses the examination of dynamical systems exhibiting chaotic and hyper chaotic behavior, which has considerable implications across several domains. In addressing these intricate systems, a primary priority is the development of strategies to successfully manage or regulate chaos. The Linear Feedback Gains Control (LFGC) framework is relevant here [28] [34].

CHAOTIC AND HYPER CHAOTIC ATTRACTORS.

A chaotic attractor in a four-dimensional fractional-order system (30) is demonstrated using the parameters $a_1 = -3, b_1 = 15, c_1 = 0.6, d_1 = -0.0001, h_1 = 1, and\ \alpha_1 = 0.95$.The outcomes are illustrated in Fig. 6. The Lyapunov exponents (LEs) $\Omega_{i,s}$ are calculated utilising Wolf's approach [31] for the specified parameter values as follows: $\Omega_1 = 0.0172, \Omega_2 = -0.0001, \Omega_3 = -0.9154, \Omega_4 = -1.1197$. If the maximum Ω_i (MLE) surpasses zero, the system transitions into a chaotic state. However, the system is hyperchaotic if it incorporates two $\Omega_{i,s} > 0$. Parameter values of $a_1 = -3, b_1 = 15, c_1 = 0.6, d_1 = -0.0001, h_1 = -1.5, and\ \alpha_1 = 0.95$ are used to generate hyperchaos in system (30). Figure 7 depicts the hyperchaotic attractor of system (30) for this parameter configuration. The values of Ωi, S are computed as follows: $\Omega_1 == 0.7861, \Omega_2 = 0.0204, \Omega_3 = = -0.0001, \Omega_4 = -3.4472$.

CHAOS REGULATION WITH A LINEAR FEEDBACK GAINS CONTROL (LFGC) FRAMEWORK.

We consider the fractional-order controlled system

$$D^{\alpha_1 - 1}\,\theta(s) = N\big(\theta(s)\big) + U\big(\theta(s)\big), \alpha_1 \in (0,1] \tag{31}$$

where $\theta(s) \in R^n$, N denotes a nonlinear vector function, and $U(\theta(s))$ represents a linear control vector function. Let bar S be an equilibrium point at the origin for the uncontrolled variant of system (31). Consequently, we present the subsequent lemma [22]:

Lemma 9. [31] If a Lyapunov function $V\big(\theta(s)\big)$ exists for the controlled system (31) with $\alpha_1 = 1$, then the equilibrium point at the origin, bar T, is at least locally asymptotically stable (LAS). $0 < \alpha_1 < 1$

For $T_0 = (0,0,0,0)$ and the positive feedback control gains (FCGs) $C_i \in R^+, i = 1,2,3,4$, denote a regulated variation of the equations. (30) is

$$\begin{aligned} D^{\alpha_1}\,x &= a_1 u - a_1 y + (h_1 - C_1)\,x - xu \\ D^{\alpha_1}\,y &= b_1 x + u - xz - C_2 \\ D^{\alpha_1}\,z &= x^2 - (C_3 + c_1)\,z \\ D^{\alpha_1}\,u &= (d_1 - C_4)\,u \end{aligned} \tag{32}$$

So, we have

Theorem 10. [26], The hyper chaotic attractors in eqns. (32) are suppressed to $T_0 = (0,0,0,0)$ provided that

$$C1 > \frac{1}{4C_2C_4 - (4d_1 + 1)}$$

$$\left\{\left(|a_1 + b_1|^2\,(C_4 - d_1) + \frac{|a_1||a_1 + b_1|}{2}\right) + \left(\frac{|a||a_1 + b_1|}{2} + |a_1|^2\,C_2\right) + \frac{\delta_y^2}{(c + C_3)}\,\left(C_2\,(C_4 - d_1) - \frac{1}{4}\right)\right\} + \delta_z + \delta_u + h_1 \tag{33}$$

$$C_2 > \frac{1}{4C_4 - 4d_1}$$

$$C_3 > -c_1, C_4 > d_1$$

where $|y| < \delta_y, |z| < \delta_z, |u| < \delta_u$.

Proof. The subsequent function is a contender for the Lyapunov function of the controlled hyper chaotic system (32) as $\alpha_1 = 1$

$$V\ (\theta_1, \theta_2, \theta_3, \theta_4) = \frac{\sum_{i=1}^{4} \theta_i^2}{2} \quad (34)$$

where θ_i refers to a state variable of eqns. (30), i.e. $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) = (x, y, z, u)$. Hence, we get

$$D^1 V = \sum_{i=1}^{4} \theta_i\ D^1\ \theta_i$$

$$= (\theta_3 - \theta_4 + h - C_1)\theta_1^2 - C_2\theta_2^2 - (C_3 + c_1)\theta_3^2 + (d_1 - C_4)\theta_4^2 + (b_1 - a_1)\theta_1\ \theta_2 - \theta_1\ \theta_2\ \theta_3 + a_1\theta_1\theta_4 + \theta_2\theta_4$$

$$\le (\delta_z + \delta_u - C_1 + h_1)|\theta_1|^2 - C_2|\theta_2|^2 - (C_3 + c)|\theta_3|^2 + (d_1 - C_4)|\theta_4|^2 + |\ b_1 + a_1||\theta_1|\ |\theta_2| + \delta_y\ |\theta_1|\ |\theta_3| + |a_1||\theta_1|\ \ |\theta_4| + |\theta_2|\ |\theta_4| = -\zeta^T M\zeta \quad (35)$$

Where

$$\zeta = [|x||y|\ |z||\ u|]^T\ , M = \begin{pmatrix} (-h - \delta_z - \delta_w - C_1) & -|\ b_1 + a_1|/2 & -\delta_y/2 & -|a_1|/2 \\ -|\ b_1 + a_1|/2 & C_2 & 0 & -1/2 \\ -\delta_y/2 & 0 & C_3 - c - 1 & 0 \\ -|a_1|/2 & -1/2 & 0 & C_4 - d_1 \end{pmatrix}$$

The Hermitian matrix M is strictly positive if all inequalities (33) are fulfilled. Consequently, it ensues that $D^1\ V\ (\theta_1, \theta_2, \theta_3, \theta_4) < 0$ for all $(\theta_1, \theta_2, \theta_3, \theta_4) \neq (0, 0, 0, 0)$ belongs to a domain $\Psi \subset C_3$ that contains a neighborhood of $(\theta_1, \theta_2, \theta_3, \theta_4) = (0, 0, 0, 0)$. Therefore, it is shown that the function V represents a Lyapunov function for the system (32) with $\alpha_1 = 1$. Consequently, according to Lemma 1, we deduce that the equilibrium point at the origin of equations (32) is at minimum locally asymptotically stable when $0 < \alpha_1 < 1$. This indicates that the hyperchaotic states of equations (30) are regulated to the origin.$T_0 = (0, 0, 0, 0)$.

The system described in equation (32) is integrated using of $a_1 = -3, b_1 = 15, c_1 = 0.6, d_1 = -0.0001, h_1 = -1.5, and\ \alpha_1 = 0.95$ and the FCGs $C_1 = 165,\ C_2 = C_3 = C_4 = 1$ which satisfy Theorem 4 . Also, according to Fig. 7, the positive bounds $\delta_y, \delta_z,\ \delta_u$ are specified as $\delta_y = 20, \delta_z = 40,\ \delta_u = 0.02$. So, Fig. 1 illustrates the excellent stabilisation outcomes.

STABILIZING $T_{1,2} = (\pm\psi, \pm\psi h_1/a_1, b_1, 0)$ VIA LFGC.

Now, suppose that $\overline{T\,'} = (\bar{t}_x, \bar{t}_y, \bar{t}_z, \bar{t}_u)$ represents the non-origin equilibrium points $T_1\ or\ T_2$. So, we use the transformation $\theta' = \theta - \overline{T\,'}$, $\theta' = (y_1, y_2, y_3, y_4)$ S to translate the point $\overline{T\,'} = (\bar{t}_x, \bar{t}_y, \bar{t}_z, \bar{t}_u)$ to the origin of coordinates.

Then a controlled version of eqns. (30) to the equilibrium point $\overline{T\,'} = (\bar{t}_x, \bar{t}_y, \bar{t}_z, \bar{t}_u)$, is introduced by

$$D^\alpha\ y_1 = a_1 y_4 - a_1 y_2 + h y_1 - y_1 y_4 + v'_1 \quad (36)$$

$$D^\alpha y_2 = y_1 + y_4 - v y_3 + v'_2$$

$$D^\alpha\ y_3 = y_1^2 - c_1 y_3 + v'_3$$

$$D^\alpha y_4 = d_1 y_4 + v'_4$$

where v'_1 , v'_2 , v'_3 , v'_4 are linear control functions given as

$$v'_1 = -a_1\bar{t}_u + a_1\bar{t}_y + \bar{t}_u y_1 + \bar{t}_x y_4 + \bar{t}_x\ \bar{t}_u - h_1\bar{t}_x - C_1 y_1 \quad (37)$$

$$v'_2 = -b_1\bar{t}_x + \bar{t}_z y_1 + \bar{t}_x y_3 + \bar{t}_x\bar{t}_z - C_2 y_2$$

$$v'_3 = c_1\bar{t}_z - 2\bar{t}_x y_1 - \bar{t}_x^2 - C_3 y_3$$

$$v'_4 = -d_1\bar{t}_u - C_4 y_4$$

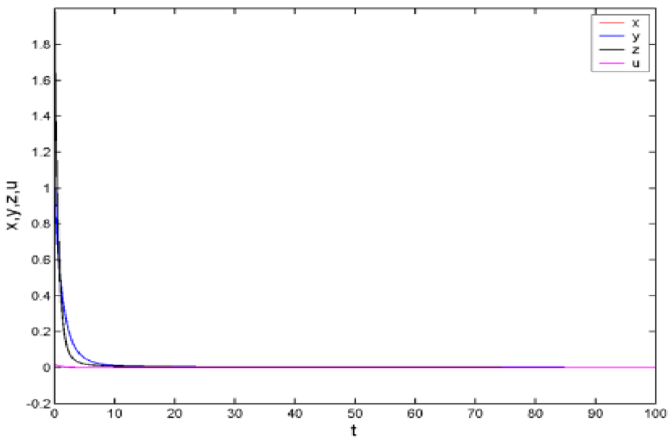

Figure 1. The trajectories of equations (32) converge to 0 with parameters $a_1 = -3, b_1 = 15, c_1 = 0.6, d_1 = -0.0001, h_1 = -1.5, and\ \alpha_1 = 0.95$ and the FCGs $C_1 = 165,\ C_2 = C_3 = C_4 = 1.$

Lemma 11. [32] The hyper chaotic states in system (36) are suppressed to $T_{1,2} = (\pm\psi, \pm\psi h_1/a_1, b_1, 0)$ if the linear controllers (37) are implemented provided that the conditions (33) hold. For the specified parameter set, FCGs and fractional order $\alpha_1 = 0.95$, the controlled system (36) undergoes numerical integration utilising the linear controllers (37). As stated in Lemma 2, all trajectories of equations (36) converge to $T_{1,2} = (\pm\psi, \pm\psi h_1/a_1, b_1, 0)$ .Thus, Fig. 2 and Fig. 3 depict the satisfactory stabilization results for T_1 and T_2, respectively.

Example 1. Examine the subsequent time-dependent system characterized by fractional order linearity, whereby $0 < \beta < 1$

$$ {}_0^C D_t^\beta\ u_1(t) = -sin2\ (s)\ u_1(s) - sin(s)\,cos(s)\ u_2(s) $$

$$ {}_0^C D_t^\beta\ u_2(t) = -sin(s)cos(s)\ u_1(s) - cos2\ (s)\ u_2(s) \tag{38} $$

Utilising the conventional Lyapunov direct method, we may ascertain the stability of system (38) by examining a quadratic function as a positive definite Lyapunov candidate.

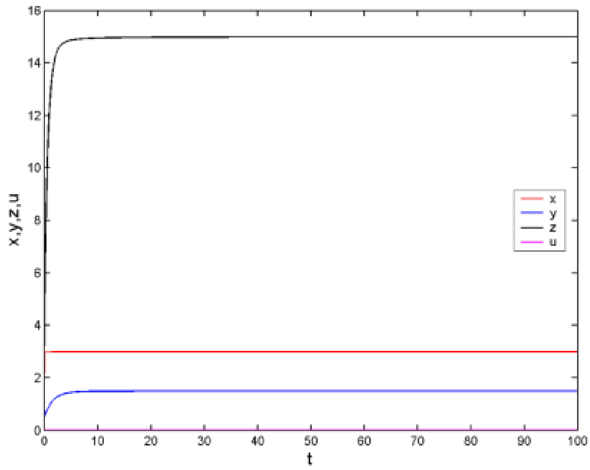

Figure 2. The trajectories of equations (36) converge to $T_1 = (\pm\psi, \pm\psi h_1/a_1, b_1, 0)$.Utilising $a_1 = -3, b_1 = 15, c_1 = 0.6, d_1 = -0.0001, h_1 = -1.5, and\ \alpha_1 = 0.95$ and the FCGs $C_1 = 165,\ C_2 = C_3 = C_4 = 1.$ and controllers ((37)) .

$$ V\left(u_1(s), u_2(s)\right) = \frac{1}{2}\ u_1^2\ (s) + \frac{1}{2} u_2^2\ (s) \tag{39} $$

By applying the fractional derivatives property described in reference [1], which asserts that $u^{\cdot}(s) = {}_0^C D_s^{1-\beta}\ {}_0^C D_s^\beta\ u(s)$, it is possible to determine that

$$ u^{\cdot}{}_1(s) = -{}_0^C D_s^{1-\beta}\ \left[u_1(s) sin^2\ (s) + u_2(s)\ sin(s)\ cos(s)\right] $$

$$u^{\cdot}{}_2(s) = -{}_0^C D_s^{1-\beta}\ [u_1(s)sin(s)\,cos(s) + u_2(s)\,cos^2\ (s)] \tag{40}$$

And then

$$\frac{dw(u_1(s),u_2(s))}{ds} = u_1(s)u^{\cdot}{}_1(s) + u_1(s)u^{\cdot}{}_2(s) =$$
$$= -u_1(s)\ {}_0^C D_s^{1-\beta}\ [u_1(s)sin^2\ (s)]u_1(s)\ {}_0^C D_s^{1-\beta}[u_2(s)\ sin(s)\ cos(s)] - {}_0^C D_s^{1-\beta}[u_2(s)\ cos^2\ (s)]$$
$$- {}_0^C D_s^{1-\beta}[u_1(s)sin(s)\ cos(s)] \tag{41}$$

As indicated by Eq. (40), it is challenging to ascertain a definitive sign for the first derivative of the Lyapunov function, hence complicating the determination of stability conclusions. Utilising the fractional-order extension of the Lyapunov direct method and presenting the Lyapunov candidate function (39), together with Property 1, it can be deduce that

$${}_0^C D_s^{-\beta}V\left(u_1(s),u_2(s)\right) = \frac{1}{2}\sum_{k=0}^{\infty}\binom{-\beta}{l}u_1^{(l)}(s)\,{}_0^C D_s^{-\beta-l}u_1(s) + \frac{1}{2}\sum_{k=0}^{\infty}\binom{-\beta}{l}u_2^{(l)}(s)\,{}_0^C D_s^{-\beta}u_2(s) \tag{42}$$

According to Eq. (42), assessing the fractional derivative of the Lyapunov function necessitates the computation of an infinite summation, including both higher-order integer and fractional derivatives of the system's states (38). This assignment is clearly challenging. Nevertheless, employing Lemma with the Lyapunov candidate function (39), it is readily established that

$${}_0^C D_s^{-\beta}V\left(u_1(s),u_2(s)\right) = \frac{1}{2}\,{}_0^C D_s^{-\beta}\ u_1^2\ (s) + \frac{1}{2}\,{}_0^C D_s^{-\beta}\ u_2^2\ (s)$$
$$\leq u_1(s)\,{}_0^C D_s^{-\beta}u_1(s) + u_2(s)\ {}_0^C D_s^{-\beta}u_2(s)) = -[u_1(s)\,sin(s) + u_2(s)\,cos(s)]^2 \leq 0 \tag{43}$$

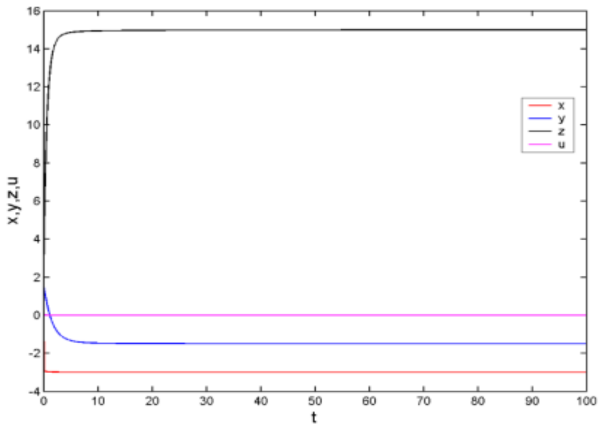

Figure 3. The trajectories of equations (36) converge to $T_2 = (-\psi, -\psi h_1/a_1, b_1, 0)$.Utilising $a_1 = -3, b_1 = 15, c_1 = 0.6, d_1 = -0.0001, h_1 = -1.5, and\ \alpha_1 = 0.95$ and the FCGs $C_1 = 165,\ C_2 = C_3 = C_4 = 1$, and controllers as specified in (37).

The negative semi-definiteness of the fractional derivative of the Lyapunov function, as seen in equation (43), indicates that the origin of the system (38) is stable. Figure 4 depicts the evolution of the system states (38) under the condition of $-\beta = 0.8$. According to the prior analytical analysis applicable to all limited initial conditions, it can be deduced that the system's origin stays stable when

$$u_1(0) = 3\ and\ u_2(0) = 6.$$

Exemple 2. Examine the fractional order nonlinear system characterised by $0 < -\beta < 1$, as represented by the equation .

$${}_0^C D_s^{-\beta}\ u_1(s) = -u_1(s) + u_2^3(s)$$
$${}_0^C D_s^{-\beta}\ u_1(s) = -u_1(s) - u_2(s) \tag{44}$$

Let we analyses the subsequent Proposed Lyapunov candidate function demonstrating positive definiteness

$$V\left(u_1(s), u_2(s)\right) = \frac{1}{2}\, u_1^2\,(s) + \frac{1}{4} u_2^4\,(s) \tag{45}$$

Now, applying Lemma , it can be found that

$$ {}_{0}^{C}D_s^{-\beta}\, w\left(u_1(s), u_2(s)\right) \;= \frac{1}{2}\, {}_{0}^{C}D_s^{-\beta}\, u_1^2\,(s) + \frac{1}{4}\, {}_{0}^{C}D_s^{-\beta}\, u_2^4\,(s) \;\le\; u_1(s)\, {}_{0}^{C}D_s^{-\beta}\, u_1(s) + \frac{1}{2}\, u_2^2\,(s)\, {}_{0}^{C}D_s^{-\beta}\, u_2^2\,(s) \le u_1(s) {}_{0}^{C}D_s^{-\beta} u_2^3\,(s) + u_1(s) {}_{0}^{C}D_s^{-\beta} u_2(s) \;\;=\; -u_1^2\,(s) - u_2^4(s) < 0 \tag{46}$$

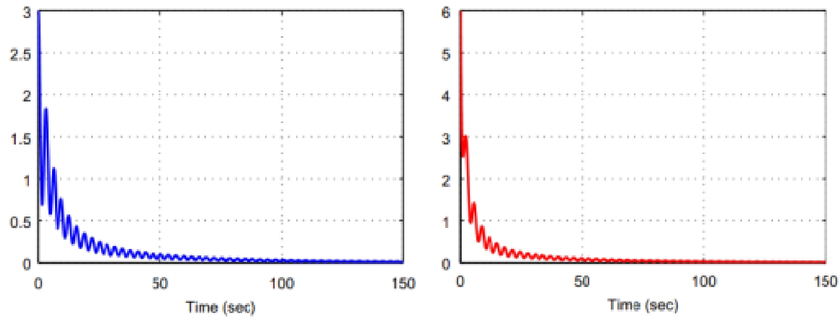

Figure 4. Evolution of the state $u_1(s)$ (left) and $u_2(s)$ (right) of the system (38), using a $\beta = 0.8$.

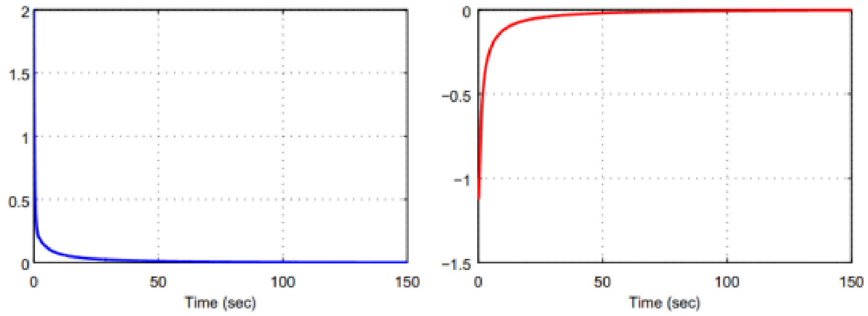

Figure 5. Evolution of the state $u_1(s)$ (left) and $u_2(s)$ (right) of the system (44), using a $\beta \;=\; 0.8$.

From the negative definiteness of the fractional derivative of the Lyapunov function in Equation (46), it can be shown from Corollary 1 that the origin of the system (44) is asymptotically stable. Figure 5 depicts the evolution of the system states with $\alpha \;=\; 0.8$. The early analytical investigation indicates that the system's origin is asymptotically stable under the limited beginning conditions of $u_1(0) \;=\; 2 \; and \; u_2(0) \;=\; -1$.

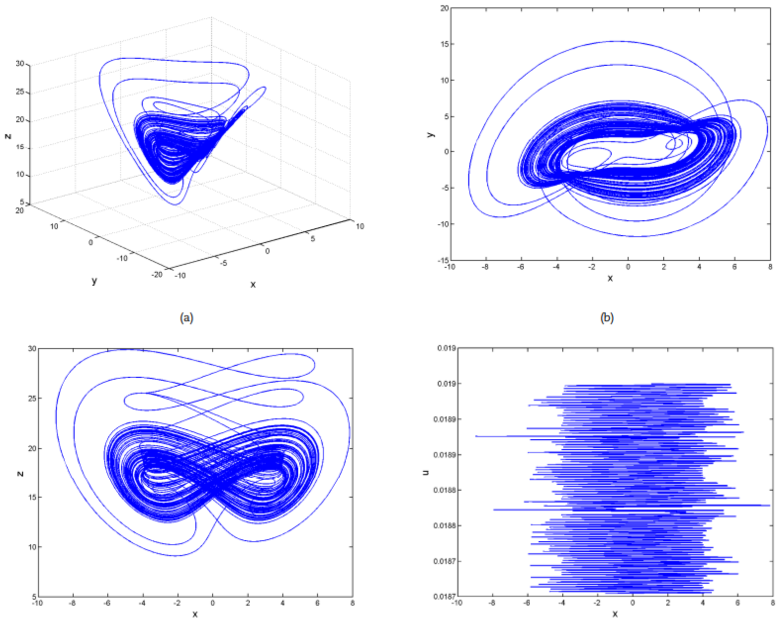

Figure 6. Chaotic attractor of system (30) is observed when $a_1 = -3, b_1 = 15, c_1 = 0.6, d_1 = -0.0001, h_1 = -1, and\ \alpha_1 = 0.95$.

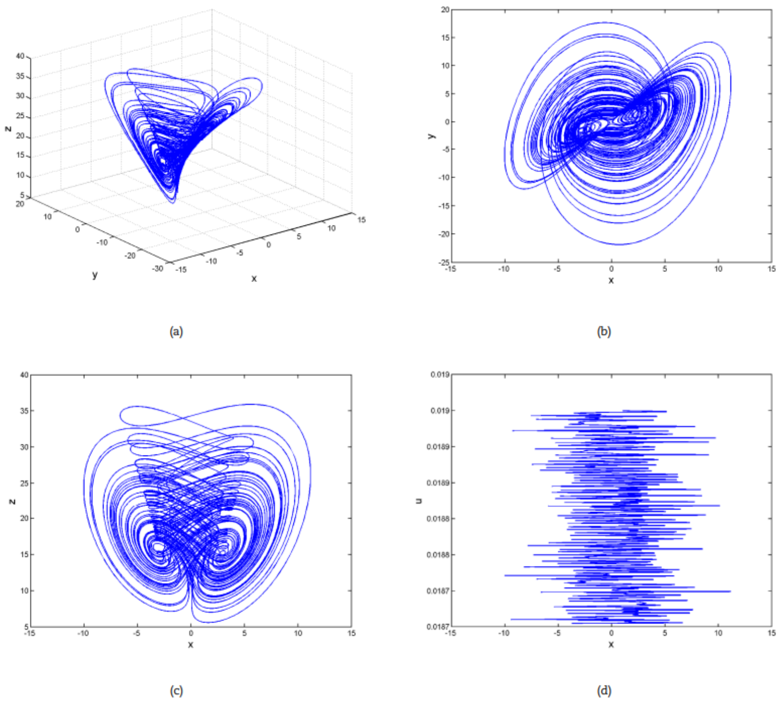

Figure 7. The hyperchaotic attractor of system (30) is observed when $a_1 = -3, b_1 = 15, c_1 = 0.6, d_1 = -0.0001, h_1 = -1.5\ and\ \alpha_1 = 0.95$

CONCLUSION

The research illustrates that controller design may effectively synchronise fractionalorder chaotic systems, hence guaranteeing safe communications and control systems. It outlines a systematic two-step controller design methodology that guarantees the accurate convergence of errors between master and slave systems to zero. The study also demonstrates the system's capacity to maintain stability under varying settings. The research illustrates successful stabilisation results for two equilibrium points, S1 and S2, employing fractional control gains (FCGs) with a fractional order of $\alpha = 0.95$[23][25]. The system's origin exhibits asymptotic stability under diverse beginning circumstances, corroborated by the negative semidefiniteness of the Lyapunov function. The results highlight the effectiveness of fractional order dynamics in controlling chaotic systems and preserving their stability [28] [30].

CONFLICT OF INTEREST

The authors declare no conflict of interest.

REFRENCES

[1] Hernandez-Castorena, G., Alcorta-Garcia, M., Saenz-Esqueda, J. & Mendez, G. Lyapunov's stability analysis for first degree polynomial systems, subject to risk-sensitive control. Mathematics And Computers In Simulation. 226 pp. 464-473 (2024,12).

[2] Fu, H., Xie, W. & Kao, Y. Adaptive sliding mode control for uncertain general fractional chaotic systems based on general Lyapunov stability. Chinese Journal Of Physics. (2024,5).

[3] Ogbumba, R., Shagari, M., Alansari, M., Khalid, T., Mohamed, E. & Bakery, A. Advancements in Hybrid Fixed Point Results and F-Contractive Operators. Symmetry. 15, 1253 (2023,6).

[4] Chen, S., Ning, C., Liu, Q. & Liu, Q. Improved multiple Lyapunov functions of input–output-to-state stability for nonlinear switched systems. Information Sciences. 608 pp. 47-62 (2022,8), olpour, R., Dehghani, M. & Talebi, H. Stability analysis of systems with time-varying delays using overlapped switching Lyapunov Krasovskii functional. Journal Of The Franklin Institute. 357, 10844-10860 (2020,10).

[5] Ait Brahim, A., El Hajaji, A., Hilal, K. & Ghordaf, J. On a Novel Fractional Calculus and its Applications to Well-known Problems. European Journal Of Pure And Applied Mathematics. 17, 1155-1167 (2024,4), http://dx.doi.org/10.29020/nybg.ejpam.v17i2.5159.

[6] Anas Faiz Alsaedy, Salah H Alshabhi, Mustafa M Mohammed, Thwiba A Khalid, Arafa O Mustafa, Runda AA Bashir, Nidal E Taha, and Awad A Bakery. Decision-making on a novel stochastic space-i of solutions for fuzzy volterra-type non-linear dynamical economic models. Contemporary Mathematics,pages 6320–6340, 2024

[7] Zhao, T., Wang, J., Lu, X. & Du, Y. Neural Lyapunov Control for Power System Transient Stability: A Deep Learning-Based Approach. IEEE Transactions On Power Systems. 37, 955-966 (2022,3).

[8] Hamadneh, T., Hioual, A., Saadeh, R., Abdoon, M., Almutairi, D., Khalid, T. & Ouannas, A. General Methods to Synchronize Fractional Discrete Reaction–Diffusion Systems Applied to the Glycolysis Model. Fractal And Fractional. 7, 828 (2023,11).

[9] Salahshour, S., Ahmadian, A., Salimi, M., Pansera, B. & Ferrara, M. A new Lyapunov stability analysis of fractional-order systems with nonsingular kernel derivative. Alexandria Engineering Journal. 59, 2985-2990 (2020,10).

[10] Michel, A. & Hou, L. Stability results involving time-averaged Lyapunov function derivatives. Nonlinear Analysis: Hybrid Systems. 3, 51-64 (2009,2).

[11] Khalid, T., Alnoor, F., Babeker, E., Ahmed, E. & Mustafa, A. Legendre Polynomials and Techniques for Collocation in the Computation of Variable-Order Fractional Advection-Dispersion Equations. International Journal Of Analysis And Applications. 22 pp. 185-185 (2024) .

[12] ˙Ilhan, E. & Kıymaz, ˙I. A generalization of truncated M-fractional derivative and applications to fractional differential equations. Applied Mathematics And Nonlinear Sciences. 5, 171-188 (2020,1).

[13] Brzezi´nski, D. Review of numerical methods for NumILPT with computational accuracy assessment for fractional calculus. Applied Mathematics And Nonlinear Sciences. 3, 487-502 (2018,7).

[14] Camlibel, M., Pang, J. & Shen, J. Lyapunov Stability of Complementarity and Extended Systems. SIAM Journal On Optimization. 17, 1056-1101 (2007,1).

[15] Devia Narrania, Kuldip Raj, Sunil K Sharma, Thwiba A Khalid, Arafa O Mustafa, Runda AA Bashir, Mustafa M Mohammed, and Awad A Bakery. Decision-making on deferred statistical convergence of measurable functions of two variables. Contemporary Mathematics, pages 5518–5533, 2024.

[16] Sun, J., Qiang, C. & Zhao, Y. Asymptotic stability for nonlinear time-varying systems on time scales. Journal Of Mathematical Analysis And Applications. 524, 127094 (2023,8).

[17] Agarwal, R., Hristova, S. & O'Regan, D. Applications of Lyapunov Functions to Caputo Fractional Differential Equations. Mathematics. 6, 229 (2018,10), http://dx.doi.org/10.3390/math6110229.

[18] Abdeljawad, T. Fractional operators with exponential kernels and a Lyapunov type inequality. Advances In Difference Equations. 2017 (2017,10).

[19] Alshammari, F., Mustafa, A., Omer, E. & Omer, F. A simulation Model of COVID-19 Epidemic Based on Vaccination and Treatment. European Journal Of Pure And Applied Mathematics. 16, 2450-2460 (2023,10), http://dx.doi.org/10.29020/nybg.ejpam.v16i4.4805.

[20] Salah H Alshabhi, OM Kalthum SK Mohamed, Mustafa M Mohammed, Thwiba A Khalid, Arafa O Mustafa, Mona Magzoub, Khdija O Taha, andAwad A Bakery. Decision-making of fredholm operator on a new variableexponents sequence space of supply fuzzy functions defined by leonardonumbers. Contemporary Mathematics, pages 5534–5545, 2024

[21] Hu, J., Lu, G., Zhang, S. & Zhao, L. Lyapunov stability theorem about fractional system without and with delay. Communications In Nonlinear Science And Numerical Simulation. 20, 905-913 (2015,3).

[22] Chen, S., Ning, C., Liu, Q. & Liu, Q. Improved multiple Lyapunov functions of input–output-to-state stability for nonlinear switched systems. Information Sciences. 608 pp. 47-62 (2022,8).

[23] Duarte-Mermoud, M., Aguila-Camacho, N., Gallegos, J. & Castro-Linares, R. Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems. Communications In Nonlinear Science And Numerical Simulation. 22, 650-659 (2015,5).

[24] Aghababa, M. A Lyapunov-based control scheme for robust stabilization of fractional chaotic systems. Nonlinear Dynamics. 78, 2129-2140 (2014,7).

[25] Chen, D., Zhang, R., Liu, X. & Ma, X. Fractional order Lyapunov stability theorem and its applications in synchronization of complex dynamical networks. Communications In Nonlinear Science And Numerical Simulation. 19, 4105-4121 (2014,12).

[26] Tareq Hamadneh, Amel Hioual, Rania Saadeh, Mohamed A Abdoon, Dalal Khalid Almutairi,Thwiba A Khalid, and Adel Ouannas. General methods to synchronize fractional discrete reaction–diffusion systems applied to the glycolysis model. Fractal and Fractional, 7(11):828, 2023.

[27] Lenka, B. TIME-VARYING LYAPUNOV FUNCTIONS AND LYAPUNOV STABILITY OF NONAUTONOMOUS FRACTIONAL ORDER SYSTEMS. International Journal Of Apllied Mathematics. 32 (2019,3).

[28] Wu, C. & Liu, X. Lyapunov and external stability of Caputo fractional order switching systems. Nonlinear Analysis: Hybrid Systems. 34 pp. 131-146 (2019,11).

[29] Owolabi, K. Modelling and simulation of a dynamical system with the Atangana-Baleanu fractional derivative. The European Physical Journal Plus. 133 (2018,1).

[30] Duarte-Mermoud, M., Aguila-Camacho, N., Gallegos, J. & Castro-Linares, R. Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems. Communications In Nonlinear Science And Numerical Simulation. 22, 650-659 (2015,5).

[31] Baleanu, D., Diethelm, K., Scalas, E. & Trujillo, J. Fractional Calculus: Models and Numerical Methods. Series On Complexity, Nonlinearity And Chaos. (2012).

[32] BUTZER, P. & WESTPHAL, U. AN INTRODUCTION TO FRACTIONAL CALCULUS. Applications Of Fractional Calculus In Physics. pp. 1-85 (2000,3).

[33] Sharma, S., Raj, K., Sharma, S., Khalid, T., Mustafa, A., Mohammed, M., Bashir, R. & Bakery, A. Applications of Strongly Deferred Weighted Convergence in the Environment of Uncertainty. International Journal Of Analysis And Applications. 22 pp. 181-181 (2024).

[34] Khalid, T., Imam, A., Bahatheg, A., Elsamani, S. & El Mukhtar, B. A fractional epidemiological model for prediction and simulation the outbreaks of dengue fever outbreaks in Sudan. Jou.