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Exact Pendant Domination in Graphs

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ABSTRACT

Let be any graph. A dominating set in is said to be an Exact dominating set if for every vertex is adjacent to exactly one vertex of and for is either adjacent to exactly one vertex of or is an isolate of . That is, for every and and for every The least cardinality of an exact dominating set in is called exact domination number, denoted by A dominating set in is called a pendant dominating set if induced sub graph of contains at least one pendant vertex. The least cardinality of a pendant dominating set in is called the pendant domination number of , denoted by . In this article, we study the exact pendant domination number for path, Complete bipartite graph, Ladder graph, Pan graph, Double star graph, Diamond graph, Fish graph and also bounds for Exact pendant domination number.

Keywords*:* Dominating set, exact dominating set, Pendant dominating set, exact pendant dominating set.

Introduction

Domination in graphs has become an important area of research in graph theory, as evidenced by the many results contained in the two books by Bondy, Haynes, Hedetniemi [2] [3] and also different kinds of domination parameters are studied with bounds on standard graphs.

Let $G = (V, E)$ be any graph with $V(G) = n$ vertices and $E(G) = m$ edges then n, m are respectively called the order and the size of the graph G. For each vertex $v \in V$, the open neighbourhood of v is the set $N(v)$ containing all the vertices of u adjacent to v and closed neighbourhood of v is the set $N[v]$ containing v and all the vertices u adjacent to v.

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. A subset D of V is a dominating set of G if every vertex of $V - S$ is adjacent to a vertex of S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating of G[4]. For a given graph $G = (V, E)$, a vertex of degree one is called pendant vertex and an edge incident to a pendant vertex is called pendant edge. Tree is a connected, undirected graph with no cycles. A dominating set S is called an exact dominating set if for every vertex $u \in S$, $|N(v) \cap S| = 1$ for every $v \in V - S$ and $|N(u) \cap S| \leq 1$ for every $u \in S$. A leaf of a tree is a vertex of degree 1, while a support vertex is a vertex adjacent to a leaf. In complete bipartite graph every vertex in one set is connected to every vertex in the other set. Ladder graph can be defined as $L_n = P_2 \times P_n$, where P_n is a path graph and it is equivalent to the $2 \times n$ grid graph. Lollipop graph is the graph obtained by

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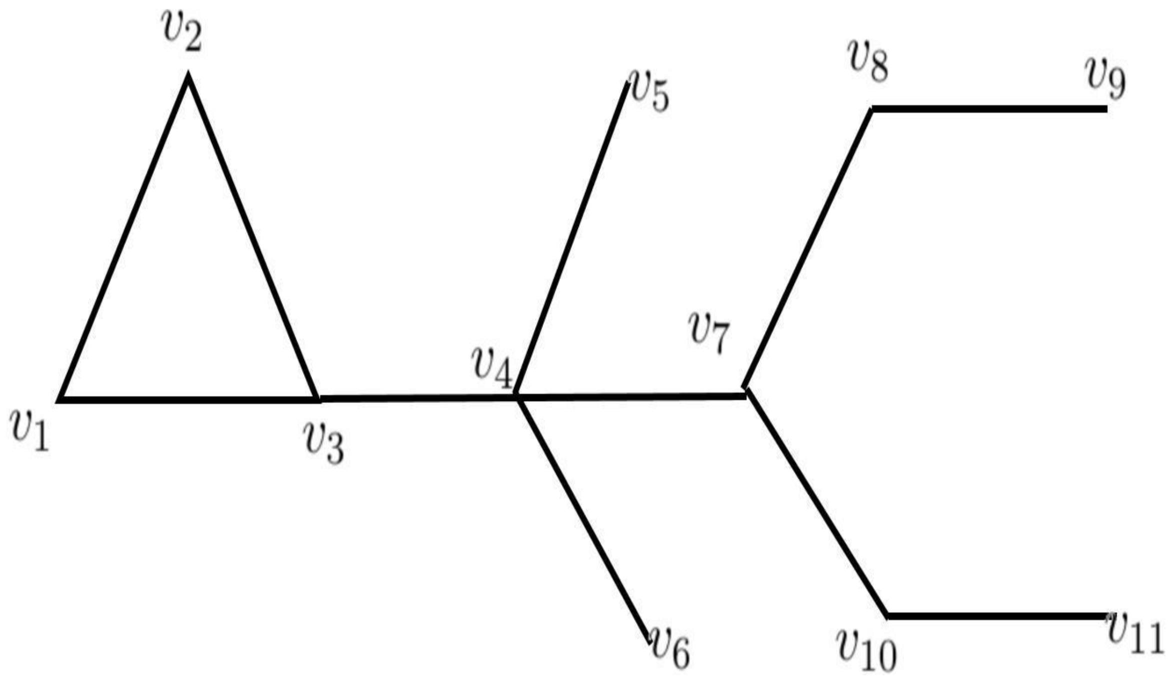
joining a complete graph $K_m$to a path graph P_n with a bridge, denoted by $L_{m,n}$. A $n -$ pan graph is the graph obtained by joining a cycle $C_n$to a singleton graph $K_1$with a bridge. A double star graph is a graph formed by joining the centres of two-star graph with a singleton edge.

A dominating set S in G is called a Pendant dominating set if the set S contains atleast one pendant vertex. The minimum cardinality of a Pendant dominating set is called the Pendant domination number $\gamma_{pe}(G)$ [5][6]. A set $S \subseteq V$ is said to be an exact dominating set if $i.$ for every $v \in V - S$ is adjacent to exactly one vertex of S or $u \in S$ is an isolate of S. That is, $|N(v) \cap S| = 1$ for every $v \in V - S$ and $|N(u) \cap S| \leq 1$ for every $u \in S$. An Exact dominating set $S \subseteq V$ is said to be a minimal dominating set if no proper subset is an exact dominating set. The exact domination number $\gamma_e(G)$ of a graph G is the cardinality of a minimum exact dominating set[1]. A dominating set S is called an Exact Pendant dominating set if $< S >$ contains at least one pendant vertex, $|N(v) \cap S| = 1$ for every $v \in V - S$ and and $|N(u) \cap S| \leq 1$ for every $u \in S$. The minimum cardinality of exact pendant dominating set is an exact pendant domination number denoted by $\gamma_{epe}(G)$.

The following is the example for a graph G having exact pendant dominating set.



The set $S = \{v_3, v_4, v_9, v_{11}\}$ is an exact pendant dominating set. The set $S = \{v_3, v_4, v_8, v_{10}\}$ is not an exact pendant dominating set because $v_7 \in V - S$ and $|N(v_7) \cap S| \neq 1$.

Observation 1.1: It is noticed that every exact pendant dominating set is a dominating set but the converse is not true. That is, every dominating set need not be an exact dominating set.

Exact Pendant Domination number for some standard graphs

Theorem 2.1: Let $G = P_n$ be a path with n vertices then

$$\gamma_{epe}(G) = \begin{cases} \frac{n}{3}+1, if\ n \equiv 0\ (mod3) \\ \left\lceil \frac{n}{3} \right\rceil, if\ n \equiv 1\ (mod3) \\ \left\lceil \frac{n}{3} \right\rceil + 1, if\ n \equiv 2(mod3) \end{cases}$$

Proof: Let $G \cong P_n$ be a path and let $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$.

We consider the following possible cases.

Case (i): Suppose $n \equiv 0(mod3)$ i.e $n = 3k$ for some integer $k > 0$ then the set $S = \{v_2, v_{3i}/1 \le i \le k\}$ is an exact pendant dominating set of G. Thus, $\gamma_{epe}(P_n) = \frac{n}{3} + 1$.

Case (ii): Suppose $n \equiv 1(mod3)$ then it is easy to check the exact pendant dominating set. The pendant dominating set itself is an exact pendant dominating set in G. Therefore, $\gamma_{epe}(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

Case (iii): Proof of this case is similar to case(i).

Theorem 2.2: Let G be a complete bipartite graph of order $2n$, then $\gamma_{epe}(G) = 2$.

Proof: Let G be a complete bipartite graph and $\{v_1, v_2, v_3, \ldots, v_{2n}\}$ are vertices of G. Let us choose the set $S = \{v_1, v_{n+1}\}$ where v_1 and v_{n+1} are two adjacent vertices in G. Then the set S will be a minimal exact pendant dominating set in G. Hence, $\gamma_{epe}(G) = 2$.

Theorem 2.3: Let G be a Ladder graph. Then the exact pendant dominating set $\gamma_{epe}(P_2 \times P_n) = 2 + \left\lceil \frac{n-1}{2} \right\rceil$.

Proof: Let G be a Ladder graph with $n \ge 4$. Fix an edge $e = \{u_2, v_2\}$ of G. Then for any dominating set of $P_2 \times P_{n-3}$ is the set $S = S' \cup \{u_2, v_2\}$ will be the minimum exact pendant dominating set of G. Where S' is the minimum dominating set of $P_2 \times P_{n-3}$. Hence, $\gamma_{epe}(P_2 \times P_n) = 2 + \left\lceil \frac{n-1}{2} \right\rceil$.

Theorem 2.4: Let G be a Lollipop graph then $\gamma_{epe}(G) = \left\lceil \frac{n}{3} \right\rceil + 2$.

Proof: Let G be a Lollipop graph of the form $L_{m,n}$ where $m = 3$ and $n = 1,2,3, \ldots$.

We have possible two cases.

Case (i): For $m = 3$ and $n = 3k$ where $k > 0$. The set $S = \{v_1, v_{3i}\}$ for $i = 1,2,3, \ldots, \left\lceil \frac{n}{3} \right\rceil + 1$ will be the minimum exact pendant dominating set.

Case (ii): For $m = 3$ and $n = 3k+1, 3k+2$ where $k > 0$. The set $S = \{v_3, v_{3i+1}\}$ for $i = 1,2,3, \ldots, \left\lceil \frac{n}{3} \right\rceil$ will be the minimum exact pendant dominating set.

Theorem 2.5: Let G be a Pan graph then $\gamma_{epe}(G) = 3 + \left\lceil \frac{n-5}{3} \right\rceil$.

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Proof: Consider a Pan graph with n vertices and the vertex set contain $S = \{v_1, v_2, v_3, \ldots, v_n, u\}$ where u is a vertex of degree one. We consider the following cases.

Case (i): Suppose $n \equiv 0(mod3)$ i.e $n = 3k$ for some integer $k > 0$. Fix the vertex of degree one and its adjacent vertex with degree three in the cycle. Remove neighbourhood vertices of degree three then the graph obtained after removing the edge and the corresponding vertices is isomorphic to P_{n-3}.

Then, the exact pendant dominating number of the Pan graph is $\gamma_{epe}(G) = 3 + \gamma(P_{n-3}) = 3 + \left\lceil \frac{n-5}{3} \right\rceil$.

Case (ii): Suppose $n \equiv 1(mod3)$. Fix the vertices v_n and v_{n-1} in the cycle and the vertex v_n is connected to the singleton set in the bridge. Remove the neighboring vertices of v_n and v_{n-1} from the pan graph then the graph obtained after removing the edges and corresponding vertices is isomorphic to P_{n-3} .

Then, the exact pendant dominating number of the Pan graph is $\gamma_{epe}(G) = 3 + \gamma(P_{n-3}) = 3 + \left\lceil \frac{n-5}{3} \right\rceil$.

Case (iii): Suppose $n \equiv 2(mod3)$ Fix the vertex of degree one in the pan graph which dominate the single vertex in the cycle C_n. Remove the vertex of degree one then the graph obtained after removing the singleton set and the corresponding vertices is isomorphic to pendant domination number of P_{n-3}.

Then, the exact pendant dominating number of the Pan graph is $\gamma_{epe}(G) = 3 + \gamma(P_{n-3}) = 3 + \left\lceil \frac{n-5}{3} \right\rceil$.

Theorem 2.6: If G is a double star graph then, $\gamma_{epe}(G) = 2$.

Proof: A double star, $S_{m,n}$consists of two star graphs $K_{1,m}$ and $K_{1,n}$whose centers are joined by an edge. Let u (center of $K_{1,m}$) and v (center of $K_{1,n}$), and their leaves $u_1, u_2,\ldots, u_m$ and $v_1, v_2,\ldots, v_n$. The set D={u,v} forms an exact dominating set for $S_{m,n}$. For every leaf u_i, it is adjacent only to u; for every leaf u_i, it is adjacent only to v; and u and v are adjacent only to each other and their respective leaves. Thus, every vertex not in D is adjacent to exactly one vertex in D.

Definition 2.1: The Diamond graph is a planar undirected graph with 4 vertices and 5 edges, specifically resembling a K_4 (complete graph of 4 vertices) with one edge removed, forming two triangles sharing an edge, and one additional edge connecting two non-adjacent vertices.

Theorem 2.7: If G is a Diamond graph then, $\gamma_{epe}(G) = 2$.

Proof: Let G be a diamond graph whose order is 4 and size 5,by taking any two adjacent vertices exact pendant domination is obtained. Thus, every vertex not in exact dominating set D is adjacent to exactly one vertex in D.

Definition 2.2: fish graph is a planar undirected graph with 6 vertices and 7 edges, formed by combining two cycles C_4 and C_3 having a vertex in common.

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Theorem 2.8: If G is a fish graph then, $\gamma_{epe}(G) = 2$.

Proof: A fish graph is a graph whose order is 6 size 7. The graph consisting of two cycles C_4 and C_3 with a common vertex. To obtain an exact pendant dominating set the common vertex and a vertex adjacent to it from C_4 can be considered which results to a minimal exact pendant dominating set with minimum cardinality $\gamma_{epe}(G) = 2$.

Definition 2.2: A pair of graphs G and H, where $O(G) = m$ and $O(H) = n$ will be considered. A graph called the corona product GoH of two graphs is created by taking one copy of G and $|V(G)| = m$ copies of H and connecting the $i-th$ vertex of G to each vertex in the $i-th$ copy of H.

Theorem 2.9: Let P_2 and P_m be two path graphs with 2 and m vertices respectively, then $\gamma_{epe}(P_2 o P_m) = 2$.

Proof: Let $V(P_n) = \{v_1, v_2\}$ and $V(P_m) = \{u_1, u_2, ..., u_m\}$ are the vertex set of the graphs P_2 and P_m respectively. The set $S = \{v_1, v_2\}$ will be the exact pendant dominating set of the graph $P_2 o P_m$. The set of vertices of the graph P_2 are dominated the vertices of P_m and each vertex in P_m are adjacent to exactly one vertex of P_2 and also the induced sub graph of S contains a pendant vertex.

Therefore, $\gamma_{epe}(P_2 o P_m) = |S| = 2$.

1. Bounds for Exact Pendant domination number $\gamma_{epe}(G)$

Theorem 3.1: Let S be a minimum exact pendant dominating set in G then $|S| \leq |V - S|$.

Proof: Let S be a minimum pendant dominating set in G. Take, $S = \{u_1, u_2, u_3, ..., u_k\}$ and $|S| = k$. Note that, every vertex of S has at least one neighbor in $V - S$. Then, k vertices of S are adjacent to at least k vertices of $V - S$. That is, $V - S \geq k = |S|$. Therefore, $|S| \leq |V - S|$.

Theorem 3.2: For any connected graph G of order n then , $\gamma_{epe}(G) \leq \frac{n}{2}$ and $\gamma_{epe}(G) \ncong P_5, C_n$.
Proof : Let, $S = \{u_1, u_2, u_3, ..., u_k\}$ be a minimum exact pendant dominating set. Then, $\gamma_{epe}(G) = k$. For n number of vertices in a graph $n = |V - S| + |S| = |V - S| + k$. Therefore, $|V - S| = n - k$. We know that, if S is a minimum exact pendant dominating set in G then $|S| \leq |V - S|$. Therefore, $|S| \leq n - k$. Since, $|S| = k$ implies $k \leq n - k$. So, $k \leq \frac{n}{2}$ yields that $\gamma_{epe}(G) \leq \frac{n}{2}$.

Theorem 3.3: For any graph G of order n we have, $2 \leq \gamma(G) \leq \gamma_{pe}(G) \leq \gamma_{epe}(G) \leq n$.

Further, $\gamma_{epe}(G) = 2$ if and only if G contains an edge of degree at least $n - 2$.

Theorem 3.4: For any graph G of order n, $\gamma_{epe}(G) + \gamma_{epe}(\bar{G}) \leq n + 2$.

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Proof: Let G be any graph. Suppose G contains an isolated vertex then $\gamma_{epe}(G) \leq n$ and $\gamma_{epe}(\bar{G}) \leq 2$. Therefore, $\gamma_{epe}(G) + \gamma_{epe}(\bar{G}) \leq n + 2$. Similarly, if $\bar{G}$ contains an isolated vertex, we obtain that, $\gamma_{epe}(G) + \gamma_{epe}(\bar{G}) \leq n + 2$.

Suppose G and $\bar{G}$ contains no isolated vertices and also, we know that if a graph G has no isolated vertices, then $\gamma_{pe}(G) \leq \frac{n}{2}$ also $\gamma_{pe}(\bar{G}) \leq \frac{n}{2}$.

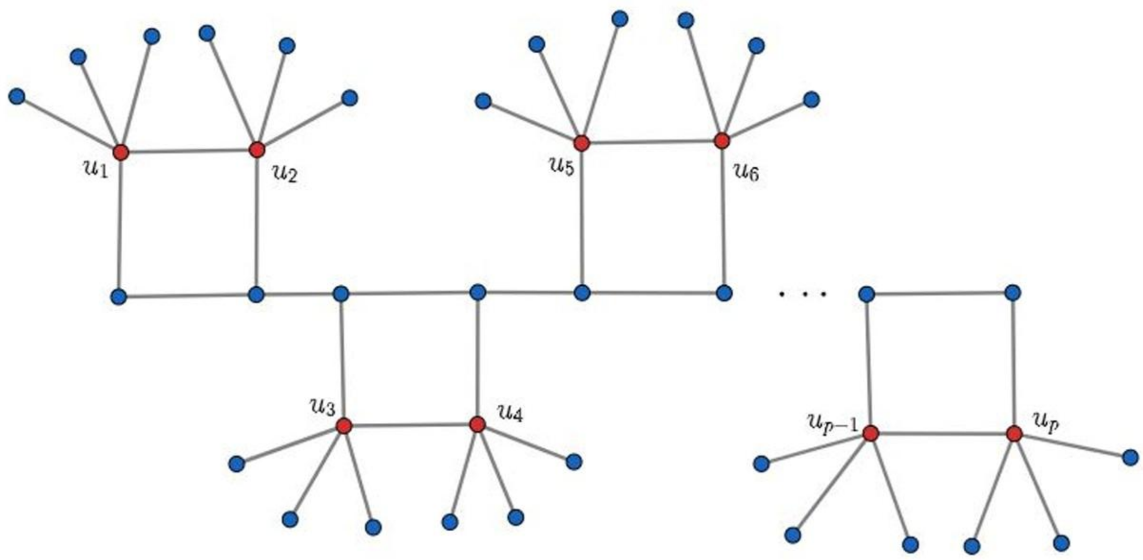
Since, $\gamma_{epe}(G) \leq \gamma_{pe}(G) + 1$ always, it follows that $\gamma_{epe}(G) \leq \frac{n}{2} + 1$ and $\gamma_{epe}(\bar{G}) \leq \frac{n}{2} + 1$. Therefore, , $\gamma_{epe}(G) + \gamma_{epe}(\bar{G}) \leq 2\left(\frac{n}{2}\right) + 2 = n + 2$. Therefore, $\gamma_{epe}(G) + \gamma_{epe}(\bar{G}) \leq n + 2$.

Theorem 3.5: For any positive integers p and q with $1 \leq p \leq q$, there exist a simple graph G such that, $\gamma_e(G) = p$ and, $\gamma_{epe}(G) = q$.

Proof: Consider the following cases.

Case (i): If $p = q$. Let G be a graph shown in the figure. clearly the set, $T = \{u_1, u_2, u_3, u_4, u_5, \ldots, u_{p-1}, \ldots, u_p\}$ is both γ_e −set and γ_{epe} −set of G.

Thus, $\gamma_e(G) = p = q = \gamma_{epe}(G)$.



Case (ii): If $p < q$. Let G be a graph shown in the figure.

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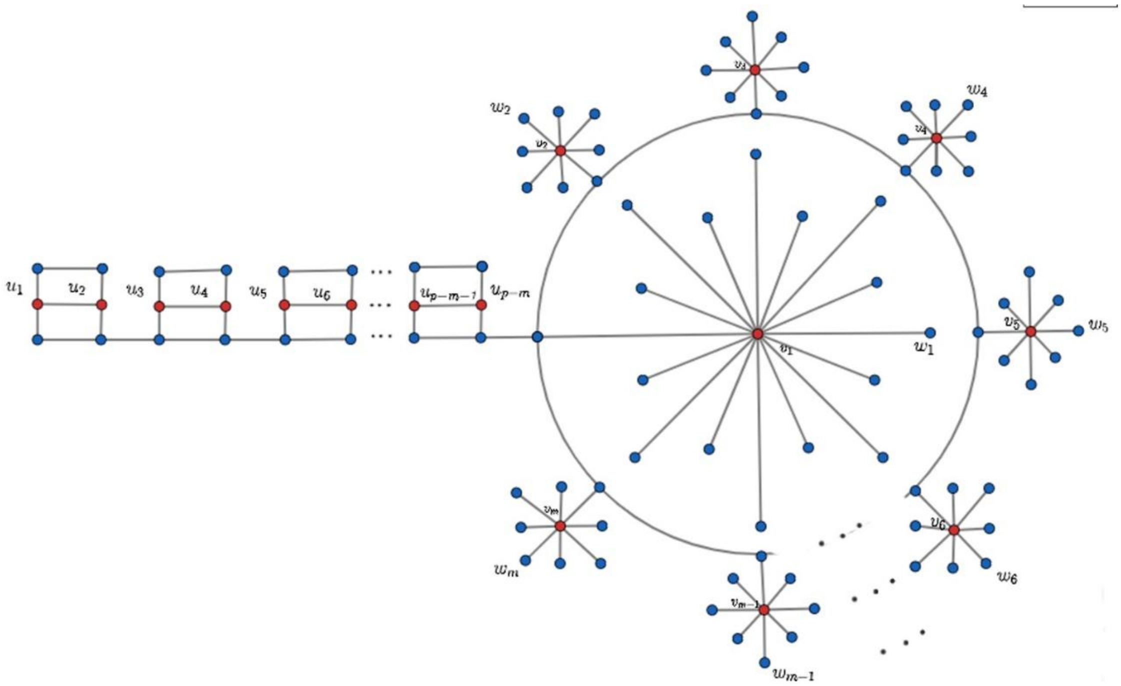

Let, $m = q - p$ and $m \in Z$. Observe that, $S_1 = \{u_1, u_2, u_3, \ldots, u_{p-m}\} \cup \{v_i; i = 1,2, \ldots, m\}$ is a γ_e −set of G and $S_2 = \{u_1, u_2, u_3, \ldots, u_{p-m}\} \cup \{v_i; i = 1,2, \ldots, m\} \cup \{w_i; i = 1,2, \ldots, m\}$ is a γ_{epe} −set of G. It follows that, $\gamma_e(G) = |S_1| = (p - m) + m = p$ and $\gamma_{epe}(G) = |S_2| = (p - m) + m + m = p + q - p = q$. Therefore, $\gamma_e(G) = p < q = \gamma_{epe}(G)$.

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