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Barycenters of Covariance Matrices and Riemannian Entropy for Analyzing Structural Risk in Financial Markets

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ABSTRACT

Introduction:

This paper offers a new geometric-entropic framework that identifies and assesses structural change points in financial markets, with a focus on the effects of the COVID-19 pandemic. Conventional econometric models continue to rely on using covariance matrices in Euclidean space, disregarding important geometric properties, such as the fact that covariance matrices are symmetric positive definite (SPD) matrices. This study emerged from the limitations of using SPD covariance matrices. The pandemic was an important systemic shock that altered correlations, volatility, and risk dependencies in the financial markets, thus allowing an ideal condition to apply and evaluate the new method.

Objectives:

The central aim is to develop and examine a new mathematical approach represent two original concepts:

• Capturing the central tendency of time-varying market covariances by locating their barycenter on the Riemannian manifold.

• Characterizing the structural disorder and uncertainty of the market using a recently introduced Riemannian entropy measure.

• Show that this dual-action approximation is superior to volatility- or correlation-based approaches in deriving timely and accurate indication of structural seams or systemic shocks, notably along with the evidence presented during the COVID-19 pandemic.

Methods:

The research employs the following methodology stages:

• Data: The analysis uses weekly log-returns for 17 major international stock market indexes ranging from January 2016 to January 2023.

• Covariance Modeling: Each rolling-window covariance matrix is estimated and treated as a point on the SPD manifold.

• Barycenter Calculation: The Riemannian barycenter for the covariance matrices is computed over time to present the evolving central dependency structure for the global markets.

• Entropy Measurement: A new Riemannian entropy measure is calculated in addition to Fréchet variance to measure disorder and dispersion of the covariance matrices about the barycenter.

• Change-Point Detection: Statistical procedures are applied to each trajectory of the barycenter and entropy to isolate distinct regime shifts associated with notable market events.

Results:

• After the outbreak of COVID-19, we observed a significant shift in market covariances' Riemannian barycenter, indicating a shift in market risk geometry. During severe market stress, especially in March 2020 and again in some of the following waves of the pandemic, we saw marked and pronounced increases in Riemannian entropy.

• We found that the increase in the entropy (a measure of structural uncertainty) occurs in advance of sudden volatility, our alerts achieve early warning of systemic stress prior to the increase in volatility identified using traditional methods.

• We found that, compared to traditional methods, whether using the barycenter or entropy on its own approach, the joint approach of barycenter movement and entropy oscillation was an effective combination of two-dimension approach, for arguably diagnosing structural breakdowns.

Conclusions:

In conclusion, we have effectively developed and applied a new geometric-entropic framework that unifies Riemannian entropy with covariance barycenters. The analysis indicates that the COVID-19 pandemic has changed both risk levels and the structural geometry of financial markets. The methodology creates an institutional formal and intuitive way to analyze dynamic financial systems, and presents a unique and worthwhile contribution to mathematical finance. The methodology has improved the early detection of structural regime shifts and financial contagion, while allowing for the study of time-varying covariances from an entirely new perspective.

Keywords: Riemannian entropy, covariance matrices, barycenters, stock market indexes, systemic risk, uncertainty.

INTRODUCTION

The financial economics field is continuously evolving, and exact measurement and interpretation of systemic risk is still a significant challenge. Traditional strategies, which are typically based on linear or Euclidean statistical perspectives, have not been effective in fully capturing the complex, nonlinear interdependencies (Bhatia, 2007) that characterize modern financial markets. Since they generally evaluate only volatility and pairwise correlations, standard models typically miss deeper structural changes within the covariance topology of the market. Consequently, they are also unable to anticipate or elucidate contagion effects, sudden shifts in correlations, and overall breakdowns in diversification during periods of crisis (Pennec, 2006).

The disruption caused by the COVID-19 pandemic greatly exemplified this shortcoming. In addition to generating large spikes in volatility, the pandemic can be characterized as a structural shock that modified the relationships among global markets and demonstrated the inadequacy of traditional risk measures in tracking changes in systemic dependencies. It is these core challenges and structural changes that have given rise to new mathematical techniques that can capture the geometry and informational intricacy of financial systems(Lang, 1999).

In reaction, the present study provides robust and unique mathematical methods to understand Riemannian Entropy and Barycenters of Covariance Matrices. Within this context, one can now talk about distances, averages, and variations in a geometrically consistent fashion by interpreting covariance matrices as points on a Riemannian manifold consisting of Symmetric Positive Definite (SPD) matrices (Bhatia & Holbrook, 2006). The Riemannian Entropy quantifies the amount of disorder and structural uncertainty around this core configuration, while the Riemannian Barycenter is a pure measure of the central tendency of market interdependence (Bhatia & Holbrook, 2006). Together, these structures produce a singular geometric-entropic representation of systemic risk that describes the structural position of the market network, potential systemic stress, and describes the overall stability of the market network.

To provide some insights into the structural geometry governing market interdependencies, how the structural geometry changed before, during, and after the COVID-19 crisis is examined using the method with empirical data from the financial market as represented by global stock indices between the years 2016-2023. The advent and propagation of systemic stress is demonstrated through change-point detection techniques to show prominent deformations of the alternating Riemannian barycenter and entropic trajectories reproduce stress. While the study

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contributes to developing a new diagnostic tool to analyze structural vulnerabilities in financial systems through a dual geometric and informational lens, this study also provides a viable avenue for the more general effort to apply differential geometry and information theory to analyze financial systems.

MATHEMATICAL PRELIMINARIES

2.1. Covariances matrices

Covariance matrices are important objects in statistics and machine learning and they are not just linear algebraic objects, rather they are elements of a specific curved space, $\mathcal{S}_{++}^{N}$, which is the Riemannian manifold of symmetric positive definite matrices (Bhatia, 2007,; Pennec, 2006). $\mathcal{S}_{++}^{N}$has a rich geometric structure and one can rigorously define distances, shortest paths (geodesics) and compute statistics that respect the geometric structure of this space (Lang, 1999; Bhatia, 2009). Following recent work in this area (Pennec, 2006; Fletcher & Joshi, 2004; Moakher, 2005), the geometric mean will be employed as a compact and useful statistic providing a summary of a set of frame covariance matrices. The Riemannian mean is an informative estimator, which is invariant to affine transformation, and has been shown to be effective with complex data (Fletcher & Joshi, 2004). For SPD matrices, A and B, the geometric mean is widely defined as the midpoint of the geodesic connecting the two.

$$A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}. \qquad (1)$$

Motivated by this geometric interpretation, Moakher (2005) generalized the definition for $k \geq 2$ matrices using a Riemannian metric. The geometric mean is defined as the unique SPD matrix X that satisfies the matrix equation:

$$\sum_{i=1}^{k} \log\left(C_i^{-1}X\right) = 0. \qquad (2)$$

This formulation generalizes the property of the geometric mean of real numbers and reduces to Eq. (1) for $k = 2$. The solution to Eq. (2) can be approximated iteratively (Pennec, 2006; Fletcher & Joshi, 2004):

$$\mu_{t+1} = \exp_{\mu_t}\left(\frac{1}{k}\sum_{i=1}^{k} \log_{\mu_t}(C_i)\right). \qquad (3)$$

where μ_0 is an initial estimate. This iteration, a gradient descent on the manifold, converges under certain conditions to the Fréchet mean. However, computing the matrix exponential and logarithm maps ($\log_{\mu_t}, \exp_{\mu_t}$), for each iteration is computationally expensive, even for symmetric matrices (Higham, 2008). This situation will compel the development of efficient, stable algorithms, especially in applications that require real-time processing such as emergency department patient flow data and financial data .

2.2. The Riemannian Manifold of $\mathcal{S}_{++}^{N}$Matrices

The space $\mathcal{S}_{++}^{N}$ is a differentiable manifold. To make it a Riemannian manifold, it is equipped with a metric. Pennec (2006) introduced the *affine-invariant* Riemannian metric. Recent work on Riemannian metric learning has extended this concept to learn optimal metrics for specific applications. For two tangent vectors A, B in the tangent space at P ($T_P\mathcal{S}_{++}^{N}$), the inner product is:

$$\langle A, B\rangle_P = \langle P^{-\frac{1}{2}}AP^{-\frac{1}{2}}, P^{-\frac{1}{2}}BP^{-\frac{1}{2}}\rangle, \qquad (4)$$

where $\langle C, D\rangle = \mathrm{tr}(C^T D)$ is the Frobenius inner product. This metric induces the norm

$$\| A \|_P = \| P^{-\frac{1}{2}}AP^{-\frac{1}{2}} \|.$$

Equipped with this metric, $\mathcal{S}_{++}^{N}$ becomes a Cartan-Hadamard manifold—a simply connected, geodesically complete manifold with non-positive curvature (Bridson & Haefliger, 1999). A key consequence is that the exponential map at any point P, $\exp_P: T_P\mathcal{S}_{++}^{N} \to \mathcal{S}_{++}^{N}$, is a global diffeomorphism. Its inverse is the logarithm map

$$\log_P: \mathcal{S}_{++}^{N} \to T_P\mathcal{S}_{++}^{N}.$$

For a point $\Sigma \in \mathcal{S}_{++}^{N}$ and a tangent vector $V \in T_\Sigma\mathcal{S}_{++}^{N}$, the geodesic starting at Σ with velocity V is:

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$$\gamma_V(t) = \Sigma^{\frac{1}{2}}\exp\left(t\Sigma^{-\frac{1}{2}}V\Sigma^{-\frac{1}{2}}\right)\Sigma^{\frac{1}{2}}.$$

Consequently, the exponential and logarithm maps are given by:

$$\exp_\Sigma(V) = \Sigma^{\frac{1}{2}}\exp\left(\Sigma^{-\frac{1}{2}}V\Sigma^{-\frac{1}{2}}\right)\Sigma^{\frac{1}{2}}, \quad (5)$$

$$\log_\Sigma(Q) = \Sigma^{\frac{1}{2}}\log\left(\Sigma^{-\frac{1}{2}}Q\Sigma^{-\frac{1}{2}}\right)\Sigma^{\frac{1}{2}}. \quad (6)$$

The functions exp and log inside these expressions are the standard matrix exponential and logarithm.

The geodesic distance between two points $P, Q \in \mathcal{S}_{++}^N$ is the length of the unique geodesic connecting them (Bhatia & Holbrook, 2006):

$$\begin{aligned}\mathrm{dist}(P,Q) &= \| \log_P(Q) \|_P \\ &= \left\| P^{\frac{1}{2}}\log\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right)P^{\frac{1}{2}}\right\|_P \\ &= \left\|\log\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right)\right\| \quad (7) \\ &= \sqrt{\mathrm{tr}\left(\left[\log\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right)\right]^2\right)}.\end{aligned}$$

2.3. The Geometric (Fréchet) Mean

Using this distance, the geometric mean of a set $\{C_i\}_{i=1}^k \subset \mathcal{S}_{++}^N$ is defined as the minimizer of the sum of squared distances:

$$\arg\min_{X\in\mathcal{S}_{++}^N} \sum_{i=1}^k \mathrm{dist}\,(X, C_i)^2 = \arg\min_{X\in\mathcal{S}_{++}^N} \rho(X). \quad (8)$$

where $\rho(X) = \frac{1}{2k}\sum_{i=1}^k \mathrm{dist}\,(X, C_i)^2$. Moakher (2005) showed that this variational problem is equivalent to the matrix equation in (2) and proved the solution is unique. This unique solution is variously termed the *Riemannian geometric mean*, *Fréchet mean*, or *Karcher mean* (Afsari, 2011).

The gradient of the cost function $\rho(X)$ is (Pennec, 2006):

$$\nabla\rho(X) = -\frac{1}{k}\sum_{i=1}^k \log_X(C_i). \quad (9)$$

This leads directly to the Riemannian gradient descent algorithm:

$$\mu_{t+1} = \exp_{\mu_t}\left(\frac{1}{k}\sum_{i=1}^k \log_{\mu_t}(C_i)\right). \quad (10)$$

Using Eqs. (4) and (5), this can be implemented as:

$$\mu_{t+1} = \mu_t^{\frac{1}{2}}\exp\left(\frac{1}{k}\sum_{i=1}^k \log\left(\mu_t^{-\frac{1}{2}}C_i\mu_t^{-\frac{1}{2}}\right)\right)\mu_t^{\frac{1}{2}}. \quad (11)$$

2.4. The Log-Euclidean Metric

As a computationally efficient alternative, Arsigny et al. (2007) proposed the *Log-Euclidean* metric. This framework endows $\mathcal{S}_{++}^N$ with a Euclidean vector space structure via the matrix logarithm. The distance between $A, B \in \mathcal{S}_{++}^N$ is:

$$\mathrm{dist}_{LE}(A,B) = \| \log(A) - \log(B) \|. \quad (12)$$

A significant advantage is that the Fréchet mean under this metric has a closed-form solution (Arsigny et al., 2007):

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$$\begin{aligned}\arg\min_{X\in\mathcal{S}_{++}^N}\frac{1}{k}\sum_{i=1}^{k}\mathrm{dist}_{LE}\,(X,C_i)^2 &= \arg\min_{X\in\mathcal{S}_{++}^N}\frac{1}{k}\sum_{i=1}^{k}\parallel\log(X)-\log(C_i)\parallel^2 \\ &= \exp\left(\frac{1}{k}\sum_{i=1}^{k}\log\,(C_i)\right).\end{aligned} \quad (13)$$

This avoids iterative schemes and is computed directly from the arithmetic mean in the tangent space at the identity, making it particularly useful in time-sensitive applications such as emergency department patient flow analysis (Higham, 2008).

In the last step, we used the fact that the first-order condition for this optimization problem is given by

$$\frac{1}{k}\sum_{i=1}^{k}\big(\log(X)-\log(C_i)\big)=0.$$

2.5. Riemannian Entropy

When conducting statistical analyses on covariance matrices, traditional Euclidean operations often fall short due to the natural Riemannian geometry of the SPD matrix space (Pennec, 2006). Based on this geometric perspective, we can define more advanced measures of dispersion that take into account the structure of manifolds (Bhatia, 2007). Riemannian Entropy is one of the most important quantities used to measure variability in this curved space, generalizing traditional concepts from statistics to data that takes values in the manifold.

This measure combines two important concepts of Riemannian geometry: the geodesic distance, which is the shortest distance between any two points on the curved manifold, and the Fréchet barycenter, a generalization of the Euclidean mean on Riemannian manifolds defined by minimizing the sum of squared geodesic distances.

For a set of covariance matrices $\{\Sigma_1,\dots,\Sigma_N\}\in P(m)$, where $P(m)$ denotes the manifold of $m\times m$ SPD matrices, the Riemannian Entropy is formally defined as:

$$H(\Sigma_1,\dots,\Sigma_N)=\frac{1}{N}\sum_{i=1}^{N}\delta_R^2\,(\Sigma_i,\bar{\Sigma})$$

Here, $\delta_R(\Sigma_i,\bar{\Sigma})$ represents the Rao-Fisher geodesic distance between the i-th covariance matrix and the Fréchet barycenter $\bar{\Sigma}$, which is itself defined as:

$$\bar{\Sigma}=\mathrm{argmin}_{\Sigma\in P(m)}\sum_{i=1}^{N}\delta_R^2\,(\Sigma_i,\Sigma) \qquad (14)$$

The geodesic distance δ_R is explicitly given by:

$$\delta_R(\Sigma_i,\bar{\Sigma})=\parallel\mathrm{Log}(\bar{\Sigma}^{-1/2}\Sigma_i\bar{\Sigma}^{-1/2})\parallel_F=\left[\sum_{j=1}^{m}\log^2\lambda_j\right]^{1/2} \qquad (15)$$

where $\mathrm{Log}(\cdot)$ denotes the matrix logarithm, $\parallel\cdot\parallel_F$ the Frobenius norm, and $\{\lambda_j\}$ the eigenvalues of $\bar{\Sigma}^{-1/2}\Sigma_i\bar{\Sigma}^{-1/2}$ (Moakher, 2005). This distance metric satisfies all metric properties while respecting the manifold's curvature, unlike Euclidean alternatives.

The Riemannian Entropy H is the direct manifold analogy of variance in the realm of Euclidean statistics (Pennec, 2006), but the manner in which it is derived fundamentally differs in that it values the non-linear curvature of the SPD manifold. Though Euclidean variance reflects average squared distances measured in a straight-line manner, Riemannian Entropy indicates the average squared distance measured along curved geodesic paths indicating a more accurate compulsiveness of dispersion on the manifold (Ying, 2019).

The interpretative value of H is profound. A minimized Riemannian Entropy indicates that covariance matrices are tightly clustered around the barycenter within the manifold's geometric structure, suggesting structural consistency and temporal stability in the underlying system. Conversely, elevated entropy values signify substantial geodesic dispersion, revealing system heterogeneity and structural instability.

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LITERATURE REVIEW

The study of nonlinear behavior in financial markets was initiated in Hsieh (1995), who recognized the existence of nonlinear dynamics and its implication in market efficiency, which set the pathway towards more complicated ways to study financial risk. From that point, research began to take on the angle of information theory, entropy, and their use in determining market disorder and inefficiency. For example, Pele et al. (2017) used entropy as a measure of market risk. Vozna (2017) presented the theoretical basis of entropy within economic analysis. Oprean et al. (2017) put forth a new methodology to measure market efficiency under a complex nonlinear dynamic paradigm. Similarly, Benedetto et al. (2021) and Brouty & Garcin (2022) utilized entropy-based methodologies to investigate information flow and market efficiency. More recent studies taken up the measure of efficiency examining time varying efficiency and financial crises Patra & Hiremath (2022), Papla & Siedleck (2024), Alkan & Süsay Alkan(2024), Vu et al. (2024) has used entropy to measure efficiency during the pandemic and Ozcan et al. (2023) have taken entropy combined with grey relational analysis to evaluate the effect of COVID-19 pandemic on financial performance.

At the same time, the use of Riemannian geometry has also expanded into the finance domain. Huang et al. (2017) used nonlinear manifold learning to create early warning signals of structural breaks in markets, while Huang et al. (2019) proposed a new method to estimate financial system fragility using manifold curvature. In addition, Abanto-Valle et al. (2021) adapted the sophisticated Riemann manifold Hamiltonian Monte Carlo methodology to model stochastic volatility in Latin American markets, while Ahn et al. (2021) brought the concept of Riemannian modeling to collective and behavioral dynamics.

Additionally, Shu et al. (2025) recently proposed a comprehensive framework that combines entropy and Riemannian geometry to understand the risk transmission processes in global stock markets, providing a link between computational and geometric strategies for financial analysis.

Following this intellectual trajectory, the current study stands out for its originality in utilizing both the entropic and Riemannian framework simultaneously to study financial markets. The development of a unified geometric-informational model provides a holistic view of structural interdependencies and provides better identification of systemic risk transition, especially during times of crises, such as during a pandemic (e.g., COVID-19).

METHODS

4.1. Data Description

This empirical research utilized a dataset of 17 prominent equity market indices, selected as representative of varying geographic and economic areas. The indices and the areas of country jurisdiction are: Europe – CAC 40 (France), DAX (Germany), Euro Stoxx 50 (Eurozone), PSI 20 (Portugal), AEX (Netherlands), and FTSE 100 (UK); the Americas – Dow Jones Industrial Average, NASDAQ Composite, S&P/TSX Composite (Canada), Bovespa Index (Brazil), and S&P/BMV IPC Index (Mexico); Asia and the Middle East – Nikkei 225 (Japan), FTSE China A50 (China), KOSPI (South Korea), Nifty 50 (India), and Tadawul All Share (Saudi Arabia); Oceania – S&P/ASX 200 (Australia).

The data consisted of weekly closing prices, with a time span from January 1, 2016 to January 1, 2023. This timespan captures market behaviour characterized by pre-COVID-19 pandemic, COVID-19 pandemic, and post-pandemic market recovery(Zaremba et al., 2020). In total, each index has 365 weekly observations. The data was collected from [https://www.investing.com/indices]. A weekly frequency was the choice in an effort to balance meaningful movements in market behavior and preventing high-frequency noise which may affect the validity of covariance matrix estimation robustness. For each index $S_i(t)$, the logarithmic return is defined as:

$$r_i(t) = \log\big(S_i(t)\big) - \log\big(S_i(t-1)\big),$$

where $r_i(t)$ represents the continuously compounded return of index i at time t . The vector of returns at time t is then:

$$R(t) = [r_1(t), r_2(t), \dots, r_{17}(t)]^\top, \qquad (16)$$

representing the joint dynamics of global equity markets.

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4.2. Rolling Covariance Matrix Time Series Construction

In order to examine the time-varying dependence across stock markets, we derived a matrix-valued time series from the data with a rolling window approach Feng, Y., Zhang, Y., & Wang, Y. (2023). Each rolling time window is 26 weeks long, which corresponds to approximately one semester. For each time point t, we consider the multivariate submatrix:

$$\mathcal{R}_t = [R(t-25), R(t-24), \dots, R(t)].$$

The sample covariance matrix is computed as:

$$\Sigma_t = \frac{1}{25}\sum_{k=0}^{25}(R(t-k) - \bar{R}_t)\,(R(t-k) - \bar{R}_t)^\top, \quad (17)$$

where $\bar{R}_t$ denotes the local mean of returns within the window. Each Σ_t is symmetric positive definite (SPD), i.e. $\Sigma_t \in \mathcal{S}_{++}^{17}$, forming a sequence:

$$\{\Sigma_t\}_{t=1}^{T} \subset \mathcal{S}_{++}^{17},$$

which constitutes a covariance matrix time series evolving on a Riemannian manifold (Pennec, 2006).

4.3. Riemannian Modeling and Geometric Framework

The space $\mathcal{S}_{++}^{n}$ of SPD matrices is a Riemannian manifold, endowed with various geometric metrics . In this work, two metrics are adopted to characterize the evolution of financial covariance structures (Bhatia, 2009).:

a) **Affine-Invariant Riemannian Metric (AIRM):**

$$\delta_R(\Sigma_1, \Sigma_2) = \| \log\left(\Sigma_1^{-1/2}\Sigma_2\Sigma_1^{-1/2}\right) \|_F,$$

This metric enables computation of Fréchet means (Riemannian barycenters) that preserve manifold structure (Pennec et al., 2019).

b) **Log-Euclidean Metric:**

$$\delta_{LE}(\Sigma_1, \Sigma_2) = \| \log(\Sigma_1) - \log(\Sigma_2) \|_F,$$

The log-Euclidean approach linearizes the SPD space by applying the matrix logarithm, facilitating computation without loss of geometric consistency (Arsigny et al., 2007) .

The Riemannian (Fréchet) mean of a set $\{\Sigma_t\}_{t\in W}$ over a time window W is defined as:

$$\Sigma_W^* = \arg\min_{\Sigma\in\mathcal{S}_{++}^{n}} \sum_{t\in W} \delta^2\,(\Sigma, \Sigma_t),$$

where $\delta(\cdot,\cdot)$ denotes either AIRM or the log-Euclidean metric. This operation provides a geometric average of covariance matrices .

4.4. Riemannian Entropy and Market Uncertainty

For the measurement of financial uncertainty we use the Riemannian entropy concept which is connected to the generalized variance of the covariance matrix (Jeuris et al., 2012). Consequently, Riemannian entropy can be written as:

$$H_R(\Sigma_t) = \frac{1}{2}\text{logdet}(2\pi e\,\Sigma_t)$$

This entropy measures the spread of market risk, which allows us to follow the changing processes of uncertainty through time. The progression of $H_R(\Sigma_t)$ and barycenters Σ_W^*, indicates three stages of uncertainty; pre-COVID (2016-2019), COVID (2020-2021) and post-COVID (2022-2023) (Shu et al., 2025)..

RESULTS AND DISCUSSION

5.1. Analysis and Interpretation of Covariance Matrices

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a. Covariance Matrix for Pre-Covid Period

The covariance matrix presented in Figure (1) shows the interrelationships among the weekly returns of some of major global stock market indices during pre-COVID-19 (2016-2019). The matrix cells demonstrate the covariance value between two financial markets, while the color gradients show the strength and direction of co-movement.

Lighter color (yellow, light green) indicates greater covariance while these two markets are moving together somewhat more than most of the other markets, which is primarily visible in those areas that are geographically and economically integrated, expect for years leading to COVID-19. An example of this includes North American markets (Dow Jones, NASDAQ, S&P/TSX) and European markets (CAC 40, DAX, FTSE 100, Euro Stoxx 50) which tend to demonstrate high regional financial and international economic integration.

Darker color values (blue and violet) represent very low or almost 0 covariance values, suggesting the markets are co-evolving largely independently from each other, which is likely more the case if the markets are geographically removed or economically segmented. This was seen, for example, between Asian indices (Nikkei, KOSPI, Nifty 50) and the Gulf market (Tadawul).

From an economic perspective, the global pattern of color shown that the structure of the global financial system was relatively diverse and segmented before the COVID-19 crisis.

Regional strength was strong, but inter-regional strength remained weak, which suggested there was a high degree of diversified global finance, and therefore, limited potential for contagion. The lighter colors along the main diagonal represent each market's own variance, representing the internal volatility of markets with intense activity such as NASDAQ and Bovespa. The prevalence of dark colors outside the diagonal shows the independence of most markets and indicates systemic global risk was low. Overall, from both a mathematical perspective and an economic perspective, the covariance matrix indicates that before the COVID-19 crisis, global financial markets were moderately connected, with strong linked primarily to regional economic clusters.

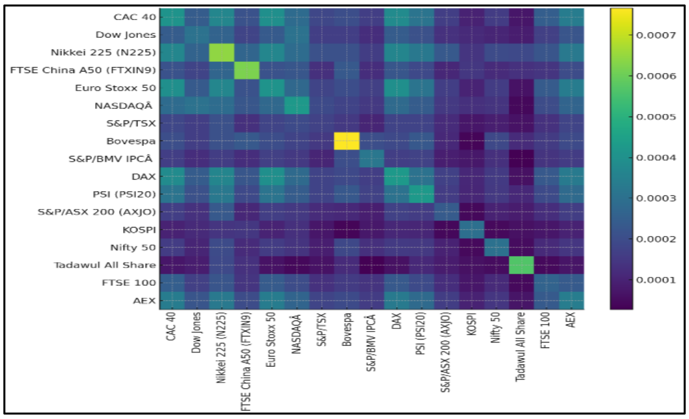

Figure (1): Covariance Matrix (Pre-COVID:01-2016 to 12-2019)

b. Covariance Matrix During Covid Period

Figure (2) shows the covariance matrix for the COVID-19 period from January 2020 to June 2021. In its mathematical nature, the matrix indicates a substantial increase in the covariance values compared to before the pandemic—this is evidenced by the greater number of lighter colors against darker colors—which indicates greater co-movements across financial markets. Lighter areas mean greater values in terms of the covariance metrics, indicating that market returns were more synchronized, implying less diversification and a greater level of systemic interdependence.

The light-colored characteristics indicated strong inter-connectedness with the major indices, which included the Dow Jones, NASDAQ, Euro Stoxx 50, DAX, and Nikkei 225, as they moved in sync signaling the systemic shock (health or economy) affecting the world, as well as global coordinated monetary and fiscal policies. On the other hand, the darker characteristics which emerged from less aligned or emerging markets (Tadawul and S&P/BMV IPC)

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suggest weaker, but still positive covariances, thus still tied to negative global effects. Even for these emerging markets, though not as integrated into the larger finance markets in the world, was still impacted and responsive to contagion effects.

From an economic point of view , the color construction in Figure (2) represents a period of increased global financial interdependence and systemic risk. The COVID-19 crisis served as a common external shock that synchronized market behaviors across regions, subsequently reducing regional uniqueness and converting the global financial system into a more uniform and interconnected structure. This improved synchronisation decreased the ability of portfolios to achieve some level of diversification across international market findings, as markets responded almost in unison to health developments, policy developments, and lockdown or reopening announcements.

Overall, the covariance matrix for the COVID-19 period clearly shows an increase in global financial synchronization, which means higher covariance values across markets, while it also clearly displayed a significant increase in systemic interconnectedness and global risk arising from a pandemic shock.

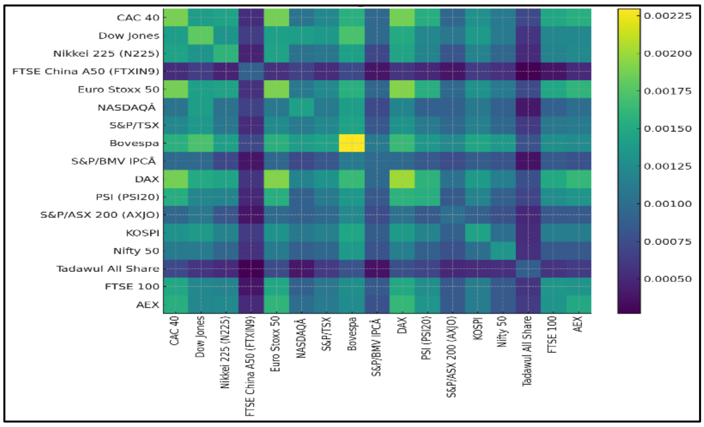

Figure (2): Covariance Matrix (During-COVID:01-2020 to 06-2021)

c. Covariance Matrix for Post Covid Period

Figure (3) presents the covariance matrix in the post-COVID-19 crisis phase, for roughly the second half of 2021 until the end of 2023. Simply put, the matrix identifies lower covariance measures from a mathematical aspect when comparing the post-crisis period to the pandemic period due to the presence of darker colors which denote generally lower covariances than the lighter colors. This still conveys the existence of less co-movements among the global financial markets overall, in addition to partially returning towards less heterogeneous, more independent markets post-pandemic, which at least in some countries featured high synchronizing behavior during the pandemic crisis.

The more notable darker areas of the matrix reflect lower covariances in co-movements across distance or structurally different economies among markets, such as Nikkei 225, Bovespa, and Tadawul, likely due to differences in recovery timing and recovery response implemented by countries impacting individual markets. The lighter matrix colours, particularly amongst the more notable major Western economies of Dow Jones, NASDAQ, Euro Stoxx 50, and DAX, indicates that some level of co-movement amongst interconnected economies continues.

In terms of economics, Figure (3) shows the phase of global financial reallocation where markets returned to a level of regional differentiation and independence in their reactions to economic shocks. This is different from during-pandemic when shocks uniformly shifted markets. Post-pandemic, markets began to move based largely on domestic fundamentals such as interest rate shifts, inflation, and speed of economic recovery. This shows that systemic risk is declining and, as correlation among markets declines, the reemergence of portfolio diversification.

The post-COVID covariance matrix restricted the global financial system from being at a high level of synchronization and shared systemic risk to an environment that becomes more stable and differentiated. Mathematically, there will be lower covariance values, and a larger area of darker colors, while economically, a gradual recovery and reestablishment of financial equilibrium across global markets are occurring.

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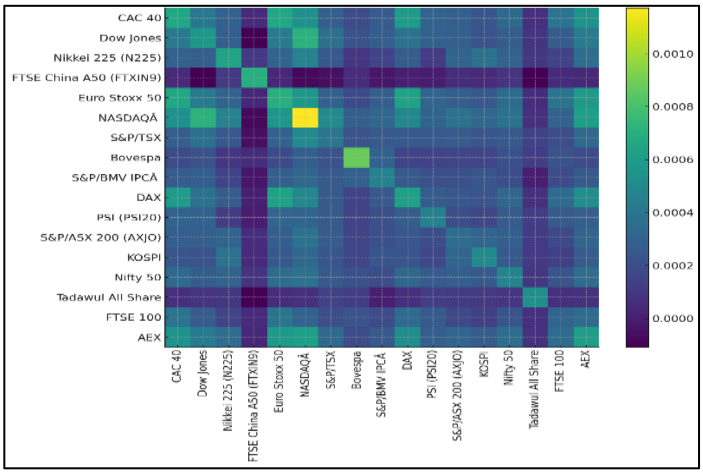

Figure (3): Covariance Matrix (Post-COVID:07-2021 to 01-2023)

5.2. Analysis and Interpretation of Riemannian Entropy

Figure 4 shows the Riemannian Entropy (RE) curve representing the time progression of structural uncertainty in the global financial system across the time span of 2016–2023. Exploring the Riemannian Entropy (RE) curve identified three phases that highlight the process of change in market complexity and interdependence.

In the pre-crisis phase (2016–2019), values for entropy remained below a range of −82 to −76, evidencing a high level of certainty and structural stability for financial markets. Within this time period, relationships among market indices were coherent and predictable. Collectively, a financial environment illustrated systemic order and homogeneity was apparent.

Towards the end of 2019 and the start of 2020, the measure of entropy fell again to approximately −79, which superficially might suggest greater certainty experienced by the markets as a result of highly synchronized financial flows. Nonetheless, the state of order and certainty of that -79 entropy is tenuous and reflects a period of structural compression and a decreasing variety of market behaviors. This entropy decline likewise illustrates the system had absorbed a degree of resilience and was increasingly exposed to systemic shocks, was a part of the early structural warning before the outbreak of the COVID-19 crisis.

The increase in entropy that was experienced during the beginning of the pandemic (2020-2021) generated around −70, constituting a large rise in structural uncertainty. Structural uncertainty increases as the previous structural arrangements break down and the global financial system becomes highly entangled and synchronized. The increase in entropy peaked again as structures transitioned from one of relative order to one of maximum structural uncertainty and systemic disorder.

After mid-2021, values of entropy began to decrease gradually and stabilized around −74 indicating a reprieve from uncertainty and an establishment of new equilibrium. However, the post-crisis equilibrium is still more complex and more uncertain than the one observed before 2019, which suggests that the financial system has adapted to a new structural base level of uncertainty.

In this context, Riemannian Entropy stands as a meaningful tool for evaluating the internal dynamics of complex financial systems. It goes beyond its conventional usage as a statistical measure of variability and becomes a basic structural metric of uncertainty, both within financial networks. It traces transitions on the path to chaos before, during, and since crisis periods. It captures how financial markets evolve from stable states, through chaos, and ultimately toward a rebalancing of complexity. High levels of entropy are indicative of increasing disorder and systemic coherence. Lower levels, by contrast, reflect increasing predictability and structure, whether real or stability that is falsely perceived. Riemannian Entropy serves as a may act as an early-warning structure, as it can evaluate depletion of certainty and systemic resiliency, and provides a strong geometric framework to analyze the tenuous balance of order and uncertainty that constitutes the global financial architecture.

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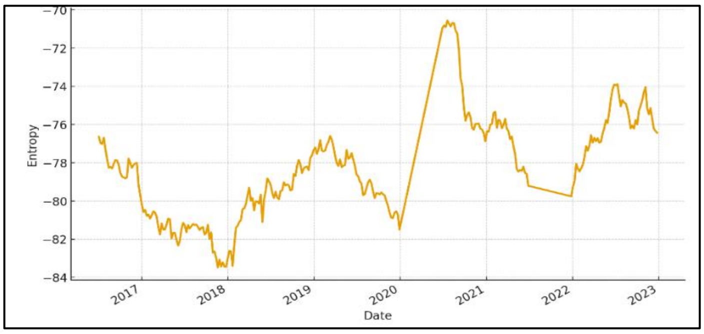

Figure (4): Riemannian Entropy (All Periods:26- Week Windows)

5.3. Analysis and Interpretation of Riemannian Distance

Figure 5 shows the trend in Riemannian Distance between the time-varying covariance matrix from financial markets and the pre-COVID reference matrix which serves as a geometric indicator of structural dissimilarity with respect to the shape of the covariance matrix and not with respect to the overall scale. A smaller distance indicates greater structural stability and some level of systemic certainty, while a greater distance implies some degree of structural divergence and added uncertainty in the financial network.

Figure 5 indicates that generally Riemannian Distance was tightly nested between 74 and 77 during the pre-COVID period (2016–2019) and represents a time of structural stability and cohesion between markets. At the start of 2020, the Riemannian Distance spiked up sharply to about 80 indicating a considerable geometric deviation from the pre-crisis baseline. This upsurge captures the systemic shock created by COVID-19 and disruption to typical correlation structures in the financial markets, instead inducing new co-movement patterns predominantly due to global uncertainty with correlated market responses.

After this peak, the distance has slowly decreased to about 75-77 by mid-2021, which indicated some structural recovery and a more complex reconfiguration of market dependencies. During the post-crisis years (2022-2023), the distance increased again to about 78, which indicates that the financial system had reached a new structural equilibrium that was quite distinct from the pre-pandemic structural framework—and was much more complex and interconnected.

Overall, Figure 5 clearly shows that the Riemannian Distance very effectively captures the geometric dynamics of structural transformation within global financial markets—taking the reader from a period of pre-crisis stability, through systemic dysfunction, to a new structural equilibrium that is characterized by considerable structural instability.

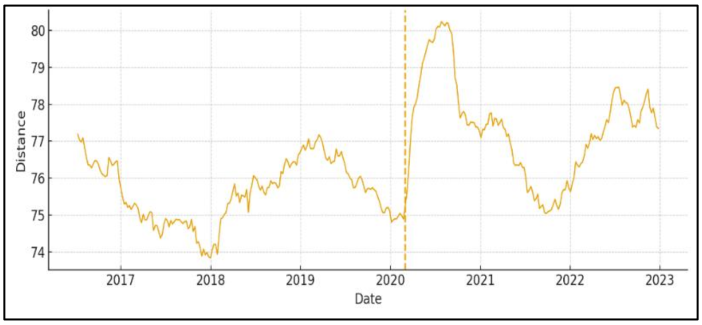

Figure (5): Riemannian Distance to Pre-Covid Barycenter (All Periods:26- Week Windows)

CONCLUSION

This study provides an explicit and comprehensive evaluation of the structural transformation in the global financial system over the pre-, during-, and post-COVID-19 phase. By using advanced mathematical and geometric tools—

specifically covariance matrices, Riemannian entropy, and Riemannian distance—the analysis tracks the evolving trajectory of global financial interconnections.

The results show a progressive trajectory: prior to 2020, the system was regionally stable with financial diversification, exemplified by lower connectivity and lower systemic risk, and then largely due to the pandemic, global market interconnectedness moved to a more aligned and highly connected global network characterized by increased systemic risk and reduced portfolios diversification opportunities. In the post-pandemic period, the system began to return to a new equilibrium, while continuing to be structurally complex and uncertain, with some regional differentiation.

The study finds a significant shift in structural regime between regional integration and global synchronisation during the crisis and the establishment of a more intricate structural equilibrium afterwards. It also notes a durable increase in structural uncertainty, relative to the pre-crisis period, and ongoing interconnectedness in the system after partial recovery. The loss of international portfolio diversification during the pandemic shows the fragility of global equity markets to shocks within the system.

Ultimately, the research illustrates how geometric and statistical tools, especially Riemannian metrics, can provide diagnostic tools for (i) systemically monitoring resilience, (ii) assess structural changes, and (iii) detect early warning indicators to in complex financial systems. It can *provide* a rigorous quantitative and geometric methodology for studying the way global capital markets go, and reorganise themselves, through cycles of stability, disruption, and reconfiguring.

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