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Research Article

Optimizing Balanced Networks with Strong Equitable Edge Colouring

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ABSTRACT

Equitable edge colouring is a sophisticated network labelling technique with no more than one difference in the number of edges between any two colour groups. In this study, we applied the equitable edge-coloring technique to triangulated networks, utilizing the minimal number of edge colours, denoted by Δ. The structural characteristics of horse stride, branch flow, and tri-wing networks are presented in this research. The technique ensures optimized network performance, regardless of whether the triangular graphs involved are in even or odd quantities, reinforcing the robustness of this method across diverse network architectures.

Keywords: Equitable edge colouring; Tri-Wing Network; Branch Flow Network; Horse Stride Network

INTRODUCTION

In graph theory, the Equitable Edge Colouring approach is a pivotal concept with far-reaching applications, particularly in networking where it plays an essential role in load balancing across multiple servers. By distributing loads evenly, equitable edge colouring has proven to enhance network performance, as demonstrated in recent studies [9]that highlight improvements in load distribution, resource allocation, and congestion management. In network contexts demanding balanced traffic flow, equitable edge colouring ensures that no communication channel (or edge) becomes disproportionately burdened. Research has delved into various graph types, such as cyclic networks and 1-planar graphs, revealing equitable colouring's potential to address scheduling complexities, optimize network flows, and facilitate team-organization scenarios. Several 2023 studies explored the equitable edge colouring of specific graph families, like splitting and wheel graphs, which serve as models for networks with dense interconnectivity [11]. These investigations underscore the criticality of minimizing edge colour imbalances at network nodes (vertices) to maintain balanced traffic loads. Recognizing that triangular networks are among the most robust, this study amalgamated triangle graphs from TW (figure 1), BFN (figure 2), and HSN (figure 3) to determine the equitable edge chromatic number. Here, vertices represent servers, and edges signify communication pathways; the core objective of this edge-colouring framework is to prevent any server from being overwhelmed by an influx of high-traffic connections of identical types.

Originally Mayer established the idea of "Equitable Edge colouring" [1].. In this concept, neighboring vertices should not have the same colour, and the count of colours in each colour class can differ by one. Hilton and de Werra [2] expanded on this idea in 1994 when they proposed the notion of equitable edge colouring. Doro [3] elaborated on this by talking about multipartite graphs that are coloured fairly. Prism equitable colouring was investigated by Sudha et al. [4], and an algorithm for determining the equitable edge chromatic number for Sunlet and Helm graph partitions was presented by Veninstine Vivik et al. [5]. It has been demonstrated by K. Kaliraj [6] that some join graphs can have evenly coloured edges.

Preliminaries

Definition:

The edges in a graph can be coloured in a special way called Equitable Edge Colouring by following some basic rules.

1) For any adjacent edges the same colour should not be assigned
2) In any 2 colour classes, the count of edges can differ at most by one

Nomenclature

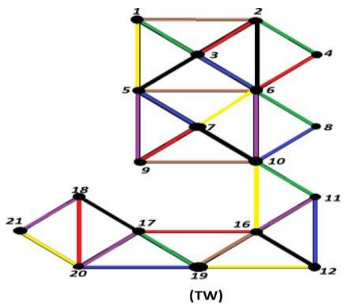
- **G=(V,E)**: A graph G has a collection of vertices (V) and edges (E).
- **Δ(G)**: Graph's peak degree.
- **χ′(G)**: The minimal number of colors required to color an edge is known as the graph G's edge chromatic number.
- **k**: The quantity of colours utilized in the colouring of an equitable edge.
- **ϕ**: An edge colouring function that applies colour to an edge.
- **A**: The network's matching number, which indicates the size of the greatest match.
- | V(G)| = n : vertex count of the given graph
- | E(G)| = m: edge count of the given graph
- **χe′(G)**: The chromatic number of an equitable edge, or the bare minimum of colours required to colour an edge equally.
- **Ci**: A group of coloured edges that have the i-th colour.
- **|Ci|**: Quantity of edges coloured with the i-th colour.
- **E(v)**: The collection of edges incident on v.
- **d(v)**: The quantity of edges adjacent to vertex v, or the degree of v.

MAIN RESULT

Within this segment, we have discussed mainly the strong equitable edge colouring graph of the Tri-Wing Network, Branch-Flow Network, and Horse Stride Network and their Equitable chromatic index.

Definition

A TriWing network (TW- figure 1) is a connected graph consisting of 17 triangular blocks joined together by edges labelled uniformly across the graph. Each block within the network is structurally identical (isomorphic). The vertex with the highest degree in this network configuration has a degree of 7.



Minimum degree = 2

Maximum degree = 7

No of vertex = 21

No of edges = 34

No of colours = Δ = 7

Figure 1. TriWing Network In Figure 1(TW), the colours red, green, blue, yellow, black, purple, and brown are denoted by the colour classes C_1, C_2, C_3, C_4, C_5, C_6, and C_7.

E(C_1) = {2- 3, 4- 6, 7- 9, 18-20, 16-17}

E(C_2) = {1- 3, 2- 4, 6- 8, 10- 11, 17- 19}

E(C_3) = {3- 6,5- 7,8-10, 11-12, 19-20}

E(C_4) = {1- 5, 6- 7, 10-16, 12-19, 20-21}

E(C_5) = {3- 5, 2- 6, 7-10, 12-16, 17-18}

E(C_6) = {5- 9, 6-10, 11-16, 17-20, 18-21)

E(C_7) = { 1- 2, 5- 6, 9-10,16-19}

All seven colours are evenly distributed among twenty-one edges.

Chromatic index $\chi'_e(\boldsymbol{TW}) = 7$

Definition

For cycle graph Cn with maximum degree Δ(G), then

$$\chi'_n(Cn) = \begin{cases} \Delta(G) + 1, & Whenever\ n\ is\ not\ even \\ \Delta(G), & Whenever\ n\ is\ even \end{cases}$$

Definition

For any triangular graph A and graph B, V(A) ∩ V(B) = 2 , E(A) ∩ E(B) = 1 and

$$V(AUB) = (V(A) + V(B)) - V(A \cap B)$$

$$E(AUB) = (E(A) + E(B)) - E(A \cap B)$$

subsequently A+ B is called the Join-graph of A and B.

Lemma

Let f be a k– AVSEEC of G

If $|E_i| = |E_j|$, i, j = 1,2,3...m, the equal number of times a colour is used , E(G) is exactly divisible by Δ(G)

f is called the Adjacent Very Strong Equitable Edge Colouring of G, and noted by k–AVSEEC of G, and

χ'_{avseec}(G) = min{k | Δ(G) = k AVSEEC of G}

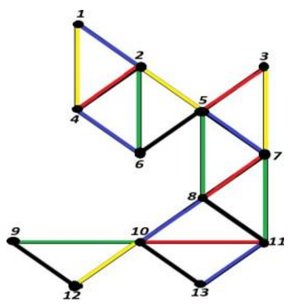
If $||E_i|-|E_j|| \leq 1$, i, j = 1, 2, 3 ...m, One more extra time few colours have to be used, It is not precisely divisible by Δ(G) in E(G). Therefore remainder < Δ(G).

f is known as an Adjacent Strong Equi- Edge Colouring of graph G, denoted by k–ASEEC(G), and

$\chi'_{asee}(G) = min\{k|\Delta(G) = k\ ASEEC\ of\ G\}$

Definition

A Branch Flow Network (BFN- figure 2) is a connected graph comprising 9 triangular blocks joined together by edges that share the same label. Each block within the network is structurally identical (isomorphic). The vertices with the highest degree in this network configuration have a degree of 5, occurring at two specific vertices, namely vertices 5 and 10.



Minimum degree = 2

Maximum degree = 5

No of vertex = 13

No of edges = 21

No of colours = Δ = 5

Figure 2. Branch Flow Network

In Figure 2(BFN), Colours red, green, blue, yellow, and black are denoted by the colour classes C_1, C_2, C_3, C_4, and C_5.

E(C_1) = {2- 4, 3- 5, 7- 8, 10-11},

E(C_2) = {2- 6, 5- 8, 7- 11, 9- 10}

E(C_3) = {1- 2, 4-6, 5-7, 8-10, 11-13}

E(C_4) = {1- 4, 2- 5, 3- 7, 10-12}

E(C_5) ={5- 6, 8-11, 10-13, 9-12}

all 5 colors are evenly distributed among twenty-one edges

Chromatic index χ'_e**(BFN)**: 5

Theorem

Every cycle in a linked network G has an equitable tricolouring as long as it is an odd cycle with C_{2x+1} in each cycle, meaning that $\Delta(G) < \chi(G)$. and the entire graph d(v)≤ $\Delta(G)$ for any n≥1, where $\Delta(G) = \chi(G)$.

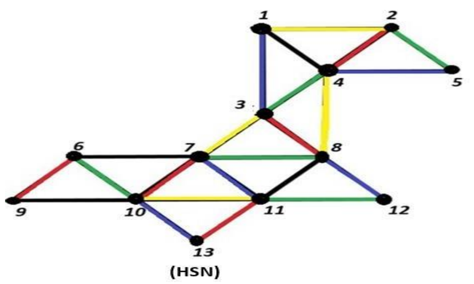
Thus, given an odd cycle C_{2x+1}, $\Delta(G)$=2, a network G (which need not be connected) fulfills $\Delta(G) \geq \chi(G)$ as long as G doesn't contain any odd cycles. In this case, $\Delta(G)$ +1 = $\chi(G)$.

Thus, in a connected graph with edge chromatic number $\Delta(G)$+1, $\Delta(G) < \chi(G)$ and each component of G when $\Delta(G) \neq 2$. Nevertheless, this does not ensure that a network G is equally $\Delta(G)$-colourable, even if $\Delta(G) \geq \chi(G)$. For some n≥1, let us examine a balanced complete bipartite graph $K_{2n+1,2n+1}$. Therefore, $2n + 1 \geq \chi(K_{2n+1,2n+1}) = 2$. It appears that $\mathbf{h}(K_{2n+1,2n+1})$ is not equally $\Delta(K_{2n+1,2n+1})$-colorable, though. In reality, the following hypothesis was put forth by Chen et al. in 1994.

The Equitable Δ-Colouring Conjecture ($E\Delta CC$),. It is equally $\Delta(G)$-colourable for a linked graph G even if every disjunct block is Δ+1. In this article, we provide the prerequisites to a network G that has $\Delta(G) \geq \chi'_e(\boldsymbol{G})$ to be fairly $\Delta(G)$-colorable.

Definition:

A Horse Stride Network (HSN figure 3) within the linked graph composed of several triangular blocks linked by edges with uniform labels. Each block in the network is structurally identical (isomorphic). The vertices with the highest degree in this configuration have a degree of 5, specifically occurring at vertices 4, 7, 8, 10, and 11.



Minimum degree = 2

Maximum degree = 5

No of vertex = 13

No of edges = 24

No of colours = Δ = 5

Figure 3. Horse Stride Network

Theorem

Regarding any natural number, the equitable chromatic index of triangular pattern is $\chi'_e\left(\boldsymbol{HSN_{1,m}}\right) = \Delta\left(\boldsymbol{HSN_{1,m}}\right)$

Proof:

Let the figure 3(horse network graph $HSN_{1,m}$) consist of 3k -2 vertices and 5k -1 edges

Let V ($HSN_{1,n}$) = $\{v_1\}$ U $\{v_n : 2 \leq n \leq 13\}$, where the vertices represented as 1, 2.... Respectively, the edges E ($HSN_{1,n}$) = $\{ e_1\} \cup \{ e_m, 2 \leq m \leq 24\}$

In this way, an edge colouring can be denoted as C: E($HSN_{1,n}$) →{ 1, 2,....}. Let's divide the edge set as follows based on the edge color classes.

In figure 3 ($HSN_{1,m}$), the colors red, green, blue, yellow, and black are represented by the letters C_1, C_2, C_3, C_4, and C_5.

E(C_1)= {2- 4, 5- 8, 6- 9, 7-10, 11-13. },

$E(C_2)$ = {2- 5, 3- 4, 7- 8, 6-10, 11-12}

$E(C_3)$= { 1- 3, 4- 5, 7-11, 8-12, 10-13}

$E(C_4)$ = { 1- 2, 4- 8, 3- 7,10-11}

$E(C_5)$ = { 1- 4, 6- 7, 8-11, 9-10}

All 5 colors are evenly distributed among 24 edges.

From the above equations, clearly the figure 3 (Horse graph $HSN_{1,m}$), is equitably edge coloured with Δ colours. Additionally we note the colour groups $E(C_1)$, $E(C_2)$, ... $E(C_\Delta)$ are distinct categories of $HSN_{1,m}$, the count of the colour groups | $E(C_1)$| = | $E(C_2)$| = | $E(C_3)$| = 5, and | $E(C_4)$| = | $E(C_5)$| = 4 , its fulfills the inequality || $E(C_i)$| - | $E(C_j)$|| ≤ 1 for i≠ j. Consequently, $\chi'_e\left(HSN_{1,m}\right) \leq \Delta$. We know that Δ = 5 and by lemma 2 $\chi'_e\left(HSN_{1,m}\right) \geq \Delta$. Therefore $\chi'_e\left(HSN_{1,m}\right) = \Delta$. Thus the equitable edge colouring 5 colours, Therefore $\chi'_e\left(HSN_{1,24}\right) \leq 5$ and the maximum degree is 5, $\chi'_e\left(HSN_{1,24}\right) \geq 5$. Hence $\chi'_e\left(HSN_{1,m}\right) = 5$.

APPLICATION

Equitable edge colouring offers a wide range of applications**.**

In this scenario, a leading UAE bank has a 1000 Mbps internet connection from Etisalat. The network is allocated among 150 servers. The various bandwidth restrictions are applied to each server based on demands, indicated by colours.

Blue: The bandwidth (BW) is limited to 200 Mbps;

Red: The bandwidth (BW) is restricted to 100 Mbps.

By allocating different bandwidths (100 or 200 Mbps), the network can optimize traffic flow among all 150 servers. This guarantees that the servers with higher demand or critical workloads receive more bandwidth, while less important servers function within their restricted bandwidth, preventing overloading and guaranteeing effective use of the entire 1000 Mbps connection.

CONCLUSION

Based on the above study and the real-world example, we have concluded that the Equitable Edge Colouring concept will be the future of networking. Herein, we determined that assigning balanced and minimum Δ colours to the Strong Equitable triangular networks will address various difficulties in the network world (including Network load balancing (NLB), distribution of resources, effective data transmission, Band width distribution). We have determined the pattern's with chromatic index. Furthermore, based on this study it is scalable to more robust and more complicated networks like cloud computing solutions can effectively use equitable edge colouring.

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this research. This study was conducted independently, without any financial or personal relationships that could influence its findings or interpretations.

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